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Existence and General Decay of Coupled Viscoelastic Waves System

Bouchelil Dounia[1], Lekdim Billal[2,3,*]

[1]Laboratory of Applied Mathematics, Ferhat Abbas Sétif 1 University, Sétif 19000, Algeria

[2]Faculty of Exact Sciences and Computer Science, University Ziane Achour of Djelfa, PO Box 3117, Djelfa, Algeria

[3]Dynamic Systems Laboratory, Faculty of Mathematics, University of Science and Technology Houari Boumediene, P.O. Box 32, El-Alia 16111, Bab Ezzouar, Algiers, Algeria

[*]Corresponding author: Lekdim Billal, b.lekdim@univ-djelfa.dz

Abstract

In this work, we study a coupled wave system with viscoelastic damping subject to Dirichlet boundary conditions on $(0,1)$. For a broad class of relaxation functions, we employ the Faedo–Galerkin method, combined with suitable a priori estimates, to establish the global existence and uniqueness of solutions. To investigate the asymptotic behavior, we use Lyapunov's method together with convexity arguments to derive general decay results. By constructing an appropriate Lyapunov functional, we show that the energy decay rate depends essentially on the properties of the relaxation function. As a consequence, classical decay rates such as exponential and polynomial decay as well as more general rates, are recovered as special cases of our approach.

Keywords

Wave system, General decay, Viscoelastic term, Global existence

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1. Introduction

Let us consider a coupled wave system motivated by viscoelastic damping:

$$\begin{cases} u_{1tt}(r,t) - \{u_{1r}(r,t) + a u_{1rt}(r,t)\}_r + b u_2(r,t) = -\int_0^t \theta(t-s) u_{1rr}(r,s)ds, & (r,t) \in (0,1)\times(0,+\infty) \\ u_{2tt}(r,t) - c u_{2rr}(r,t) + b u_1(r,t) = -c\int_0^t \theta(t-s) u_{2rr}(r,s)ds, & (r,t) \in (0,1)\times(0,+\infty) \\ u_1(0,t) = u_2(0,t) = 0, \quad u_1(1,t) = u_2(1,t) = 0, & \forall t > 0, \\ u_1(x,0) = u_{10}(x), \quad u_{1t}(x,0) = u_{11}(x) & \forall x \in (0,1) \\ u_2(x,0) = u_{20}(x), \quad u_{2t}(x,0) = u_{21}(x) & \forall x \in (0,1) \end{cases} \tag{1}$$

Here, θ is a relaxation function, and $(u_{10}, u_{11}, u_{20}, u_{21})$ denote the initial data. We assume that $u_{10}, u_{20} \in H_0^1(0,1)$ and $u_{11}, u_{21} \in L^2(0,1)$. The positive constants a, b and c represent the Kelvin-Voigt damping coefficient, the coupling coefficient, and the wave speed of the second component, respectively.

The subscripts t and r denote partial derivatives with respect to time and space, respectively; that is,

$$u_t = \frac{\partial u}{\partial t}, \quad u_r = \frac{\partial u}{\partial r}, \quad u_{rt} = \frac{\partial^2 u}{\partial t \partial r}, \quad u_{tt} = \frac{\partial u}{\partial t^2}, \quad u_{rr} = \frac{\partial u}{\partial r^2}.$$

The convolution (memory) term appearing in the system represents the viscoelastic damping and accounts for the history of the material, where the associated kernel is the relaxation function [1,2]. For problems involving non-linear viscoelastic term, we mention [3,4].

This variation in the system equations is essential, as the model can be interpreted as describing transverse vibrations of a cable in two distinct directions. In this framework, the wave speed associated with each component depends on the direction of propagation, which explains the difference between the two wave speeds.

Moreover, the wave speed influences the memory boundary term, and therefore explicitly appears in its formulation. The dissipation, on the other hand, arises from directional asymmetry. For instance, external effects such as gravity may act more strongly in one direction than in the other, leading to unequal damping mechanisms.

More generally, this formulation can also represent a bidirectional coupling between two media, such as two connected layers or interacting fields that exchange energy.

Wave propagation arises when a localized disturbance in a continuous medium is transmitted to adjacent regions through internal restoring forces. This physical mechanism is mathematically described by the classical wave equation, which models vibrations in elastic media such as strings, membranes, and solids. In many practical applications ranging from mechanical engineering to material science the primary objective is not only to describe wave propagation but also to control vibrations and analyze the long-time behavior and stability of such systems.

To achieve stabilization, various damping mechanisms have been introduced and extensively studied. Among them, Kelvin-Voigt damping and viscous damping have attracted considerable attention due to their strong dissipative effects and their ability to generate exponential or polynomial decay of energy (see, e.g., [5-13]). In contrast, viscoelastic damping, which incorporates memory effects through convolution kernels, generally produces weaker dissipation and requires more delicate analytical techniques to establish stability results (see [14-22]). For wave equations with nonlinear damping mechanisms, more complex dynamical behaviors may arise, including singular effects and intricate decay patterns, as discussed in [23,24].

An important question in control theory concerns whether localized damping is sufficient to stabilize the entire system. In this direction, Liu and Zhang [25] investigated a string equation with local Kelvin–Voigt damping of the form

$$u_{tt} = \left[u_r + b(r)u_{rt}\right]_r,$$

where $b \in L^\infty(0,1)$. They showed that the associated semigroup exhibits either polynomial or exponential stability depending on the properties of the coefficient $b(r)$, thereby highlighting the decisive role of the damping distribution.

Coupled wave systems have also been widely investigated due to their relevance in modeling interacting physical fields. Hassine and Souayeh [26] analyzed the coupled system

$$\begin{cases} u_{tt} = \{u_r + b(r)u_{rt}\}_r - v_t, \\ v_{tt} = c v_{rr} - u_t, \end{cases}$$

with $c > 0$ and, $b \in L^{\infty}(0,1)$, proving strong stability of solutions. Their results demonstrate how coupling effects may compensate for partial dissipation.

In the framework of viscoelasticity, coupled systems involving the elasticity operator

$$\Delta_e u = \mu \Delta u + (\lambda + \mu)\nabla(\mathrm{div} u)$$

have been studied to understand the interplay between elastic structure and memory effects. Beniani et al. [27] established exponential decay for such systems, while Taouaf et al. [28] obtained similar results under additional structural assumptions involving memory reduction in part of the system. On the other hand, Ming et al. [29] proved finite-time blow-up results for certain damped coupled system of semi-linear wave equations, revealing the delicate balance between dissipation and source effects, see also [30-32]. Aounallah et al. [33] investigated systems governed solely by viscoelastic damping and derived decay estimates that were later generalized and refined in subsequent works, leading to broader classes of relaxation functions and more flexible decay frameworks. Additional developments in this direction can be found in [34-42].

Despite these significant advances, several aspects remain insufficiently explored, particularly for coupled wave systems involving finite memory terms under general relaxation kernels. Many existing results either impose restrictive assumptions on the kernel or focus on specific decay types (exponential or polynomial), leaving room for a more unified treatment that accommodates broader classes of memory functions.

Motivated by these observations, the present study investigates system (1) with a finite memory term under minimal structural assumptions on the relaxation kernel. By considering a wide class of admissible kernel functions, we derive general decay results that extend and improve several recent contributions in the literature. Our approach provides a unified stability framework capable of capturing different decay rates within a single analytical setting.

2. Methods and Materials

This section introduces the preliminary concepts and tools required to establish the main results.

We first fix the notation used throughout the paper. Let $t \geq 0$ denote the time variable and $r \in (0,1)$ the spatial variable. We consider the standard Lebesgue space, $L^2(0,1)$, endowed with the norm $\|\cdot\|$, and the Sobolev space $H_0^1(0,1)$.

We define the functional

$$(\theta \circ u)(t) = \int_0^t \theta(t-s)\, \| u(t) - u(s) \|^2 \, ds.$$

The kernel θ is assumed to satisfy the following conditions:

(H1) Let $\theta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be a decreasing function that satisfies

$$\int_0^{\infty} \theta(s) ds = \overline{\theta} < 1,$$

such that

$$1 - \overline{\theta} > b,\ 1 - \overline{\theta} > b/c \ \text{ and } \ \theta(0) > 2 \max\left\{ \frac{\beta(1 - \overline{\theta} - b)}{b^2 \overline{\theta}},\ \frac{1 - \overline{\theta} - b/c}{(1 - \overline{\theta})^2} \right\} > 0,$$

for some $\beta \in (0,1)$.

(H2) We assume the existence of a positive C^1 function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ that is linear or a strictly increasing and strictly convex C^2 function on $(0, l]$, moreover $l \leq \theta(0)$, and , $\Theta(0) = \Theta'(0) = 0$, and a positive differentiable and non-increasing function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ so that

$$\theta'(t) \leq -\zeta(t)\Theta(\theta(t)) , \quad \forall t \geq 0.$$

Remark 1 ([43]). Since θ is a positive and decreasing function, we have $\theta(t) \geq 0$ for all $t \in [0, t_0]$, and

$$0 < \theta(t_0) \leq \theta(t) \leq \theta(0) , \ \forall t \in [0, t_0]$$

Moreover, since Θ and $\zeta(t)$ are positive and continuous functions, their product is also positive and continuous on $[0, t_0]$. Hence, there exists a constant $d \geq 0$ such that

$$\zeta(t)\Theta(\theta(t)) \geq d , \ \forall t \in [0, t_0]$$

Consequently, there exist two positive constants t_0 and d such that

$$\theta'(t) \le -\zeta(t)\Theta(\theta(t)) \le -d \le -\frac{d}{\theta(0)}\theta(t), \quad \forall t \in [0, t_0]. \tag{2}$$

The energy corresponding to problem (1) is expressed as

$$E(t) = \frac{1}{2}\left(\| u_{1t} \|^2 + \| u_{2t} \|^2\right) + \frac{1 - \int_0^t \theta(s)ds}{2}\left[\| u_{1r} \|^2 + c\ \| u_{2r} \|^2\right] + \frac{1}{2}\left(\theta \circ u_{1r} + c\,\theta \circ u_{2r}\right) + b\int_0^1 u_2 u_1. \tag{3}$$

Remark 2. It should be noted that this expression differs from the classical energy (kinetic and potential). It is modified to incorporate the memory term appearing in the system and can be regarded as a Lyapunov functional candidate. The classical energy is recovered by taking the relaxation function to be identically zero.

Proposition 3. *Under the hypotheses **(H1)**, we get*

$$E(t) \ge 0, \quad \forall t \ge 0.$$

Proof. Utilizing Young's and Poincare's inequalities, we deduce

$$b\int_0^1 u_1 u_2 dr \ge \frac{-b}{2}\| u_2 \|^2 - \frac{b}{2}\| u_1 \|^2 \ge \frac{b}{2}\| u_{2r} \|^2 - \frac{b}{2}\| u_{1r} \|^2 .$$

The energy will be

$$E(t) \ge \frac{1}{2}\left(\| u_{1t} \|^2 + \| u_{2t} \|^2\right) + \frac{1 - \int_0^t \theta(s)ds - b}{2}\| u_{1r} \|^2 + \frac{\left(1 - \int_0^t \theta(s)ds\right)c - b}{2}\| u_{2r} \|^2 + \frac{1}{2}\left(\theta \circ u_{1r} + c\,\theta \circ u_{2r}\right)$$

Hypothesis (***H1***) ensures that $1 - \bar{\theta} > b$, where $\bar{\theta} = \int_0^\infty \theta(s)ds.$ Since

$$\int_0^t \theta(s)ds \le \bar{\theta}, \quad \forall t \ge 0,$$

it follows that

$$1 - \int_0^t \theta(s)ds \ge 1 - \bar{\theta} > b.$$

Similarly, we also have

$$1 - \int_0^t \theta(s)ds > \frac{c}{b}.$$

Therefore,

$$1 - \int_0^t \theta(s)ds - b \ge 0 \ \text{ and } \ c\left(1 - \int_0^t \theta(s)ds\right) - b \ge 0.$$

Consequently, we obtain $E(t) \ge 0.$

Lemma 4. *The energy* (3) *satisfies*

$$E'(t) = -a\| u_{1rt} \|^2 - \frac{1}{2}\theta(t)\left[\| u_{1r} \|^2 + c\| u_{2r} \|^2\right] + \frac{1}{2}\left[\theta \circ u_{1r} + c\theta' \circ u_{2r}\right] < 0. \tag{4}$$

Proof. We multiply $(1)_1$ by u_{1t} and $(1)_2$ by u_{2t}, then integrate over $(0,1)$, and summing the resulting equations, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\| u_{1t} \|^2 + \| u_{2t} \|^2 + \| u_{1r} \|^2 + c\| u_{2r} \|^2 + 2b\int_0^1 u_1 u_2\right] \\ &= -a\| u_{1rt} \|^2 + \int_0^t \theta(t-s)\int_0^1 u_{1r}u_{1rt}drds + c\int_0^t \theta(t-s)\int_0^1 u_{2r}u_{2rt}drds.\end{aligned}$$

For the convolution terms, we'll employ this identity

$$\int_0^t \theta(t-s)\int_0^1 u_{1r}u_{1rt}drds = -\frac{1}{2}\theta(t)\| u_{1r} \|^2 + \frac{1}{2}\theta' \ u_{1r} - \frac{1}{2}\frac{d}{dt}\left[(\theta \circ u_{1r}) - \int_0^t \theta(s)ds\ \| u_{1r} \|^2\right]$$

to get (4).

Remark 5. *Based on **(H1)**, **(H2)** and Lemma* 4*, we can infer energy decay. Furthermore*,

$$E(0) \ge E(t) \quad \forall t > 0.$$

3. Well-posedness

This part focuses on establishing the existence result for the problem described in (1). To accomplish this, we will utilize the Faedo-Galerkin approach.

Theorem 6. *Assume* ***(H1)*** *and* ***(H2)*** *holds. For any initial data* $(u_{10},u_{11})\in H_0^1(0,1)\times L^2(0,1)$ *and* $(u_{20},u_{21})\in H_0^1(0,1)\times L^2(0,1)$. *Then there exists a unique weak solution to problem (1) satisfying*

$$u_1\in L^\infty((0,\infty);\, H_0^1(0,1)),\quad u_{1t}\in L^\infty((0,\infty);\, L^2(0,1))\bigcap L^2((0,\infty);\, H_0^1(0,1)),$$

And

$$u_2\in L^\infty((0,\infty);\, H_0^1(0,1)),\quad u_{2t}\in L^\infty((0,\infty);\, L^2(0,1)).$$

Proof. we use Faedo-Galerkin method. Let us consider $\{\omega_i\}_{i=1}^\infty$ and $\{z_i\}_{i=1}^\infty$ as bases of the space $H_0^1(0,1)$. Then we define the finite-dimensional subspaces $V_m=\text{span}\{\omega_1,\omega_2,...,\omega_m\}$ and $Z_m=span\{z_1,z_2,...,z_m\}$. The initial conditions are projected on these subspaces V_m and Z_m as follows

$$u_{10}^m(r)=\sum_{i=1}^m a_i\omega_i\ ,\ u_{11}^m(r)=\sum_{i=1}^m b_i\omega_i \ \text{ and }\ u_{20}^m(r)=\sum_{i=1}^m c_iz_i\ ,\ u_{21}^m(r)=\sum_{i=1}^m d_iz_i$$

such that

$$\begin{cases}(u_{10}^m,u_{20}^m)\to(u_{10},u_{20}) & \text{in } H_0^1(0,1)\times H_0^1(0,1),\\ (u_{11}^m,u_{21}^m)\to(u_{11},u_{21}) & \text{in } L^2(0,1)\times L^2(0,1).\end{cases}$$

Now, we search the approximate solutions

$$u_1^m(r,t)=\sum\nolimits_{i=1}^m\phi_i^m(t)w_i(r)\ \text{ and }\ u_2^m(r,t)=\sum\nolimits_{i=1}^m g_i^m(t)z_i(r)$$

to the finite dimensional Cauchy problem

$$\begin{cases}\int_0^1 u_{1tt}^m w dr+\int_0^1 u_{1r}^m w_r dr+a\int_0^1 u_{1rt}^m w_r dr+b\int_0^1 u_2^m w dr=\int_0^t\theta(t-s)\int_0^1 u_{1r}^m\,\omega_r drds\,,\\ \int_0^1 u_{2tt}^m z dr+c\int_0^1 u_{2r}^m z_r dr+b\int_0^1 u_1^m z dr=\int_0^t\theta(t-s)\int_0^1 u_{2r}^m z_r drds\,,\\ (u_1^m(0),\,u_2^m(0))=(u_{10}^m,\,u_{20}^m),\ (u_{1t}^m(0),u_{2t}^m(0))=(u_{11}^m,\,u_{21}^m)\end{cases}\tag{5}$$

According to the theory of ODE, the problem (5) has solutions $(\phi_i^m(t),\, g_i^m(t))$ on $[0,t_m]$. The following a priori estimates prove that $t_m=+\infty$.

Energy estimates: Multiplying $(5)_1$ and $(5)_2$ by $(\phi_i^m(t))'$ and $(g_i^m(t))'$ respectively, and then combining them, we infer

$$\frac{d}{dt}E^m(t)+\|u_{1rt}^m\|^2+\frac{1}{2}\theta(t)[\|u_{1r}^m\|^2+c\|u_{2r}^m\|^2]=\frac{1}{2}(\theta'\circ u_{1r}^m+c\theta'\circ u_{2r}^m)\le 0\ ,\tag{6}$$

where

$$E^m(t)=\frac{1}{2}\Big(\|u_{1t}^m\|^2+\|u_{2t}^m\|^2\Big)+\frac{1-\int_0^t\theta(s)ds}{2}\Big[\|u_{1r}^m\|^2+c\|u_{2r}^m\|^2\Big]+\frac{1}{2}(\theta\circ u_{1r}^m+c\,\theta\circ u_{2r}^m)+b\int_0^1 u_2^m u_1^m\ge 0.$$

As $u_{10}^m,\,u_{20}^m$ and $u_{11}^m,\,u_{21}^m$ are bounded in $H_0^1(0,1)$ and $L^2(0,1)$. Integrate (6) over $(0,t)$ yields

$$E^m(t)+\int_0^t\|u_{1rt}^m\|^2+\frac{1}{2}\theta(t)[\|u_{1r}^m\|^2+c\|u_{2r}^m\|^2]ds\le\varepsilon^m(0)=e_0\ ,$$

where $e_0>0$ that does not depend on t and m . Consequently

$$\begin{cases}(u_1^m),\ (u_2^m)\ \text{bounded in} & L^\infty(0,\infty;\, H_0^1(0,1)),\\ (u_{1t}^m),\ \text{bounded in} & L^\infty(0,\infty;\, L^2(0,1))\bigcap L^2(\,0,\infty; H_0^1(0,1),\\ (u_{2t}^m),\ \text{bounded in} & L^\infty(0,\infty;\, L^2(0,1)).\end{cases}$$

Hence, we may extract a subsequence (u_1^m) and (u_2^m) denoted by (u_1^m) and (u_2^m) so that

$$\begin{cases} u_1^m \rightharpoonup^* u_1 & \text{in } L^\infty(0,\infty; H_0^1(0,1)), \\ u_{1t}^m \rightharpoonup^* u_{1t} & \text{in } L^\infty(0,\infty; L^2(0,1)) \cap L^2(0,\infty; H_0^1(0,1)), \\ u_2^m \rightharpoonup^* u_2 & \text{in } L^\infty(0,\infty; H_0^1(0,1)), \\ u_{2t}^m \rightharpoonup^* u_{2t} & \text{in } L^\infty(0,\infty; L^2(0,1)). \end{cases}$$

Consequently,

$$\begin{cases} u_1^m \rightharpoonup u_1 & \text{in } L^2(0,\infty; H_0^1(0,1)), \\ u_{1t}^m \rightharpoonup u_{1\mathrm{t}} & \text{in } L^2(0,\infty; H_0^1(0,1)), \\ u_2^m \rightharpoonup u_2 & \text{in } L^2(0,\infty; H_0^1(0,1)), \\ u_{2t}^m \rightharpoonup u_{2t} & \text{in } L^2(0,\infty; H_0^1(0,1)). \end{cases}$$

Taking the limit in the approximate problem (5), yields a weak solution of (1) (see [44-47]).

Uniqueness: Let $U_i = u_i - \tilde{u}_i$, $i = 1,2$, be the difference of two weak solutions. Then U_1, U_2 satisfies

$$\begin{cases} U_{1tt} - U_{1rr} - aU_{1rrt} + bU_2 = -\int_0^t \theta(t-s) U_{1rr}(r,s) ds, \\ U_{2tt} - cU_{2rr} + bU_1 = -c\int_0^t \theta(t-s) U_{2rr}(r,s) ds, \end{cases}$$

with Dirichlet homogeneous boundary conditions and initial data

$$U_i(\cdot,0) = 0, \quad U_{it}(\cdot,0) = 0, \quad i = 1,2.$$

Multiply the first equation by U_{1t} and the second by U_{2t} , then integrate over $(0,1)$. Using standard arguments (as in Lemma 4), we obtain

$$E_U^{'}(t) = -a \| U_{1rt} \|^2 - \frac{1}{2}\theta(t)[\| U_{1r} \|^2 + c \| U_{2r} \|^2] + \frac{1}{2}[\theta' \circ U1r + c\theta' \circ U_{2r}] < 0,$$

where

$$E_U(t) = \frac{1}{2}\left(\| U_{1t} \|^2 + \| U_{2t} \|^2\right) + \frac{1 - \int_0^t \theta(s) ds}{2}\left[\| U_{1r} \|^2 + c \| U_{2r} \|^2\right] + \frac{1}{2}(\theta \circ U_{1r} + c\,\theta \circ U_{2r}) + b\int_0^1 U_2 U_1.$$

Since the initial data are zero, we have $E_U(0)$ By monotonicity and positivity of $E_U(t)$ it follows that

$$0 \le E_U(t) \le E_U(0) = 0 \quad \forall t \ge 0.$$

Hence, $U_1 = U_2 = 0$ Therefore, the solution is unique.

4. Decay Result

Our main objective is to prove the stability of the system's solutions by demonstrating the decay of the energy over time. To achieve this, we employ a carefully constructed Lyapunov functional alongside energy estimates to analyze the system's behavior.

Lemma 7. *Under the assumption (**H1**), for any* $\beta_1 > 0$ *, the functional*

$$M_1(t) = \int_0^1 u_{1t} u_1 dr + \int_0^1 u_{2t} u_2 dr + \frac{a}{2}\| u_{1r} \|^2,$$

satisfies

$$M_t^{'}(t) \le \| u_{1t} \|^2 + \| u_{2t} \|^2 - \left(1 - \bar{\theta} - \frac{\beta_1}{2} - b\right)\| u_{1r} \|^2 - \left[c(1 - \bar{\theta} - \frac{\beta_1}{2}) - b\right]\| u_{2r} \|^2 + \frac{c_\beta}{2\beta_1}\left[(L \circ u_{1r}) + c(L \circ u_{2r})\right] \tag{7}$$

where $\beta 1$ *is a very small positive number, and* $L(t) = \beta\theta(t) - \theta'(t), \quad 0 < \beta < 1$ and $C_\beta = \int_0^\infty \frac{\theta^2(s)}{\beta\theta(s) - \theta'(s)} ds.$

Proof. Direct computation, utilizing (1), and Young's inequality, yields

$$M_t'(t) = \| u_{1t} \|^2 + \| u_{2t} \|^2 - \| u_{1r} \|^2 - c \| u_{2r} \|^2 - 2b \int_0^1 u_1 u_2 dr \\ + \int_0^1 u_{1r} \int_0^t \theta(t-s)\, u_{1r}(s) ds dr + c\int_0^1 u_{2r} \int_0^t \theta(t-s)\, u_{2r}(s) ds dr \tag{8}$$

From Young's and Poincare's inequalities, then for any $\beta > 0$,

$$-2b\int_0^1 u_1 u_2 dr \le b \| u_1 \|^2 + b \| u_2 \|^2 \le b \| u_{1r} \|^2 + b \| u_{2r} \|^2 .$$

For the penultimate term

$$\int_0^1 u_{1r}(t) \int_0^t \theta(t-s) u_{1r}(s) ds dr = \int_0^1 u_{1r}(t) \int_0^t \theta(t-s)(u_{1r}(s) - u_{1r}(t)) ds dr + (\int_0^t \theta(s) ds) \ \| u_{1r} \|^2 \\ \le \frac{1}{2\beta_1} \int_0^1 (\int_0^t \theta(t-s) | u_{1r}(s) - u_{1r}(t) | ds)^2 dr + \frac{\beta_1 + 2\theta}{2} \| u_{1r} \|^2 . \tag{9}$$

And taking into consideration

$$\int_0^1 \left(\int_0^t \theta(t-s) | u_{1r}(s) - u_{1r}(t) | ds \right)^2 dr \\ \le \int_0^1 \left(\int_0^t \frac{\theta(t-s)}{\sqrt{\beta\theta(t-s) - \theta'(t-s)}} \sqrt{\beta\theta(t-s) - \theta'(t-s)} \, | u_{1r}(s) - u_{1r}(t) | ds \right)^2 dr \\ = \left(\int_0^t \frac{\theta^2(t-s)}{\beta\theta(t-s) - \theta'(t-s)} ds \right) \int_0^1 \int_0^t [\beta\theta(t-s) - \theta'(t-s)] \, | u_{1r}(s) - u_{1r}(t) |^2 ds dr \\ \le C_\beta (L \circ u_{1r}) \tag{10}$$

The last term in (8) can be treated in the same way as (9). Now, by combining all the above estimates, we obtain (7).

Lemma 8. *Assuming (H1), the functional*

$$M_2(t) = -\int_0^1 u_{2t} \int_0^t \theta(t-s)(u_2(t) - u_2(s)) ds dr$$

satisfies

$$M_2'(t) \le -\left(\int_0^t \theta(s) ds - \frac{\beta_2}{2}\theta(0) \right) \| u_{2t} \|^2 + \frac{b_2 \bar{\theta}}{2\beta} \beta_2 \| u_{1r} \|^2 + \frac{c(1-\bar{\theta})^2}{2} \beta_2 \| u_{2r} \|^2 + \left[\frac{1}{2\beta_2} + \left(\frac{c}{2\beta_2} + c \right) C\beta \right] (L \circ u_{2r}), \tag{11}$$

where β_2 *is a very small positive number.*

Proof. We start by differentiating $M_2(t)$, we get

$$M_2'(t) = -\int_0^1 u_{2tt} \int_0^t \theta(t-s)(u_2(t) - u_2(s)) ds dr - \int_0^1 u_{2t} \int_0^t \theta'(t-s)(u_2(t) - u_2(s)) ds dr - \int_0^1 u_{2t} \int_0^t \theta(t-s) u_{2t}(t) ds dr,$$

by exploiting (1) into $M_2(t)$ after integration by parts, we find

$$M_2'(t) = -\left(\int_0^t \theta(s) ds \right) \| u_{2t} \|^2 - \int_0^1 u_{2t} \int_0^t \theta'(t-s)(u_2(t) - u_2(s)) ds dr + b\int_0^1 u_1 \int_0^t \theta(t-s)(u_2(t) - u_2(s)) ds dr \\ + c\int_0^1 u_{2r} \int_0^t \theta(t-s)(u_{2r}(t) - u_{2r}(s)) ds dr - c\int_0^1 \left(\int_0^t \theta(t-s) u_{2r}(s) ds \right) \int_0^t \theta(t-s)(u_{2r}(t) - u_{2r}(s)) ds dr.$$

Through the application of Young's and Poincare's inequalities, Hence

$$-\int_0^1 u_{2t} \int_0^t \theta'(t-s)(u_2(t) - u_2(s)) ds dr + b\int_0^1 u_1 \int_0^t \theta(t-s)(u_2(t) - u_2(s)) ds dr \\ \le \frac{\beta_2}{2}\theta(0) \| u_{2t} \|^2 - \frac{1}{2\beta_2}(\theta' \circ u_{2r}) + \frac{\beta_2 b^2}{2\beta} \bar{\theta} \| u_{1r} \|^2 + \frac{\beta}{2\beta_2}(\theta' \circ u_{2r}) \\ = \frac{\beta_2}{2}\theta(0) \| u_{2t} \|^2 + \frac{\beta_2 b^2}{2\beta} \bar{\theta} \| u_{1r} \|^2 + \frac{1}{2\beta_2}(L \circ u_{2r}).$$

By Young's inequality and similarly to (10), we obtain

$$
\begin{aligned}
& c\int_0^1\left(u_{2r}-\int_0^t\theta(t-s)u_{2r}(s)ds\right)\int_0^t\theta(t-s)(u_{2r}(t)-u_{2r}(s))dsdr \\
& = c\left(1-\int_0^t\theta(s)ds\right)\int_0^1 u_{2r}\int_0^t(t-s)(u_{2r}(t)-u_{2r}(s))dsdr + c\int_0^1\left(\int_0^t\theta(t-s)(u_{2r}(t)-u_{2r}(s))ds\right)^2 dr \\
& \le \beta_2\frac{c(1-\bar{\theta})^2}{2}\|u_{2r}\|^2+\left(\frac{c}{2\beta_2}+c\right)\int_0^1\left(\int_0^t\theta(t-s)(u_{2r}(t)-u_{2r}(s))ds\right)^2 dr \\
& \le \beta_2\frac{c(1-\bar{\theta})^2}{2}\|u_{2r}\|^2+\left(\frac{c}{2\beta_2}+c\right)C_\beta(L\circ u_{2r}).
\end{aligned}
$$

From the above estimates, it follows that (11) holds.

Lemma 9. *Assuming (H1), the functional*

$$F(t)=E(t)+N_1M_1(t)+N_2M_2(t),$$

satisfies, for some positive constants $\delta_1,\delta_2,k_1,k_2$,

$$\delta_1 E(t)\le F(t)\le \delta_2 E(t),\quad \forall t\ge 0, \tag{12}$$

and

$$F'(t)\le -kE(t)+k_1[(\theta\circ u_{1r})+c(\theta\circ u_{2r})]. \tag{13}$$

Proof. Using Young's and Poincaré's inequalities, we obtain

$$|M_1(t)|\le\|u_{1t}\|^2+\frac{1+a}{2}\|u_{1r}\|^2+\frac{1}{2}\|u_{2r}\|^2,$$

and

$$
\begin{aligned}
|M_2(t)| & \le\frac{1}{2}\|u_{2t}\|^2+\frac{1}{2}\int_0^1\left(\int_0^t\theta(t-s)(u_2(t)-u_2(s))ds\right)^2 dx \\
& \le\frac{1}{2}\|u_{2t}\|^2+\bar{\theta}\int_0^t\theta(t-s)\|u_{2r}(t)-u_{2r}(s)\|^2 ds\le\frac{1}{2}\|u_{2t}\|^2+\bar{\theta}(\theta\circ u_{2r})(t).
\end{aligned}
$$

Consequently, there exists a constant $\delta>0$ such that

$$|N_1M_1(t)+N_2M_2(t)|\le\max\left\{N_1+\frac{1}{2}N_2,\ \frac{1+a}{2}N_1,\ \bar{\theta}N_2\right\}E(t)=\delta E(t),\ t\ge 0.$$

By choosing N_1 and N_2 sufficiently small so that $\delta_1<1$, we obtain (12) with $\delta_1=1-\delta>0$ and $\delta_2=1+\delta$.

To prove (13), we apply Lemmas 4, 7, and 8

$$
\begin{aligned}
F'(t) & \le -a\|u_{1rt}\|^2-\frac{1}{2}\theta(t)\left(\|u_{1r}\|^2+c\|u_{2r})\|^2\right)+\frac{1}{2}(\theta'\circ u_{1r}+c\theta'\circ u_{2r})+N_1\left(\|u_{1t}\|^2+\|u_{2t}\|^2\right) \\
& -N_1(1-\bar{\theta}-\beta_1/2-b)\|u_{1r}\|^2-N_1[c(1-\bar{\theta}-\beta_1/2)-b]\|u_{2r})\|^2 \\
& +\frac{c_\beta N_1}{2\beta_1}[(L\circ u_{1r})+c(L\circ u_{2r})]+N_2(-\left(\int_0^t\theta(s)ds-\beta_2/2\theta(0)\right)\|u_{2t}\|^2 \\
& +\frac{b_2\bar{\theta}}{2\beta}\beta_2\|u_{1r}\|^2+\frac{c(1-\bar{\theta})^2}{2}\beta_2\|u_{2r}\|^2+\left[\frac{1}{2\beta_2}+\left(\frac{c}{2\beta_2}+c\right)C_\beta\right](L\circ u_{2r}),
\end{aligned}
$$

let $t_0>0$ be fixed and $\theta_0=\int_0^t\theta(s)ds>0,$ using Poincare's inequality ($\|u_{1t}\|^2\le\|u_{1rt}\|^2$) and taking, $\beta_2=\dfrac{\theta_0}{\theta(0)},\ N_1<\dfrac{a}{2},\ \beta_1=\min\{1-\bar{\theta}-b,\ 1-\bar{\theta}-b/c\},$ for $t\ge t_0$, we are able to obtain

$$
\begin{aligned}
F'(t) & \le -\frac{a}{2}\|u_{1t}\|^2-\left[\frac{\theta_0}{2}N_2-N_1\right]\|u_{2t}\|^2-\left[\frac{1-\bar{\theta}-b}{2}N_1-\frac{b^2\bar{\theta}}{2\beta}\frac{\theta_0}{\theta(0)}N_2\right]\|u_{1r}\|^2 \\
& -\left[\frac{1-\bar{\theta}-b/c}{2}N_1-\frac{(1-\bar{\theta})^2}{2}\frac{\theta_0}{\theta(0)}N_2\right]c\|u_{2r}\|^2+\frac{1}{2}[\theta\circ u_{1r}+c\theta\circ u_{2r}] \\
& +\left[\frac{C_\beta N_1}{2\beta_1}+\left(\frac{1}{2\beta_2}+\left(\frac{c}{2\beta_2}+c\right)C_\beta\right)N_2\right][(L\circ u_{1r})+c(L\circ u_{2r})].
\end{aligned}
$$

For the coefficient of the second term to be negative, it is required that

$$N_1 < \frac{\theta_0}{2} N_2.$$

For the coefficients of the third and fourth terms to be negative, the following conditions must hold:

$$\frac{b^2\bar{\theta}}{\beta(1-\bar{\theta}-b)}\frac{\theta_0}{\theta(0)}N_1 < N_2 \text{ and } \frac{(1-\bar{\theta})^2}{1-\bar{\theta}-b/c}\frac{\theta_0}{\theta(0)}N_1 < N_2.$$

Combining these constraints, we obtain

$$\max\left\{\frac{b^2\theta}{\beta(1-\bar{\theta}-b)}, \frac{(1-\bar{\theta})^2}{1-\bar{\theta}-b/c}\right\}\frac{1}{\theta(0)}N_2 < N_1 < \frac{1}{2}N_2.$$

This interval is nonempty if

$$\max\left\{\frac{b^2\theta}{\beta(1-\bar{\theta}-b)}, \frac{(1-\bar{\theta})^2}{1-\bar{\theta}-b/c}\right\}\frac{1}{\theta(0)} < \frac{1}{2}.$$

According to hypothesis (**H1**), this condition is satisfied.

Now, we choose N_1 and N_2 small enough such that

$$\frac{C_\beta N_1}{2\beta_1} + \left(\frac{1}{2\beta_2} + \left(\frac{c}{2\beta_2} + C\right)C_\beta\right)N_2 \le \frac{1}{2}.$$

then, for some $k > 0$, we have (13).

Theorem 10. *Assume that (**H1**) and (**H2**) hold. Then there exist positive constants n_1 and n_2 such that the energy satisfies*

$$E(t) \le n_2\Theta_1^{-1}\left(n_1\int_{\Theta^{-1}(l)}^{t}\zeta(s)ds\right), \quad \forall t \ge t_0, \tag{14}$$

where Θ_1 is convex and strictly decreasing on $(0,l]$, it is defined as follows

$$\Theta_1(t) = \int_t^l \frac{1}{s\Theta'(s)}ds.$$

Proof. We first estimate the memory term on $[0,t_0]$ using (3) and (4), we obtain, for $i = 1,2$,

$$\int_0^{t_0}\theta(s)\,\|u_{ir}(t)-u_{ir}(t-s)\|^2\,ds \le -\frac{\theta(0)}{d}\int_0^{t_0}\theta'(s)\,\|u_{ir}(t)-u_{ir}(t-s)\|^2\,ds \le -\mu_i E'(t)$$

where $\mu_i > 0,\ i = 1,2$ are positive constants.

From these last two results, we obtain

$$\int_0^{t_0}\theta(s)\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds + \int_0^{t_0}\theta(s)\,\|u_{2r}(t)-u_{2r}(t-s)\|^2\,ds \le -mE'(t)$$

here $m = \mu_1 + \mu_2$.

Using Lemma 9, for all $t > t_0$ and for some, $k, k_1 > 0$,

$$\begin{aligned} F'(t) &\le -kE(t) + k_1[(\theta \circ u_{1r})] + c(\theta \circ u_{2r})] \\ &\le -kE(t) - mE'(t) + k\int_{t_0}^{t}\theta(s)\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds + k_1\int_{t_0}^{t}\theta(s)\,\|u_{2r}(t)-u_{2r}(t-s)\|^2\,ds. \end{aligned}$$

Taking $\mathcal{L}(t) = mE(t) + F(t)$, we have

$$\mathcal{L}'(t) \le -kE(t) + k\int_{t_0}^{t}\theta(s)[\|u_{1r}(t)-u_{1r}(t-s)\|^2 + \|u_{2r}(t)-u_{2r}(t-s)\|^2]ds. \tag{15}$$

According to the properties of Θ, we analyze two cases:

1) $\Theta(t)$ **linear:** Using (**H2**) and (4), we have

$$\begin{aligned}\zeta(t)\mathcal{L}'(t) &\le -k\zeta(t)E(t) + k_1\zeta(t)\int_{t_0}^{t}\theta(s)\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds + k_1\int_{t_0}^{t}\theta(s)\,\|u_{2r}(t)-u_{2r}(t-s)\|^2\,ds\\ &\le -k\zeta(t)E(t) + k_1\int_{0}^{t}\zeta(s)\theta(s)\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds + k_1\int_{t_0}^{t}\zeta(s)\theta(s)\,\|u_{2r}(t)-u_{2r}(t-s)\|^2\,ds\\ &\le -k\zeta(t)E(t) - k_2E'(t),\end{aligned}$$

since ζ decreasing, we have $\zeta(t)\le\zeta(s)$ for $s\le t$, so we can replace $\zeta(t)$ by $\zeta(s)$ inside the integrals. Then the integrals become exactly $-\theta'(s)$ after using (**H2**) (since $\theta'(s)\le-\zeta(s)\Theta(\theta(s))$ and for linear Θ we have $\Theta(\theta(s))=c_0\theta(s)$. Finally, using (5) again we obtain $k_2E'(t)$, then

$$(\zeta\mathcal{L}+CE)'(t)\le -k\zeta(t)E(t),$$

and $\zeta\mathcal{L}+CE\sim E$. From Gronwall's lemma, we find

$$E(t)\le k_2e^{-k\int_{t_0}^{t}\zeta(s)ds}.$$

2) $\Theta(t)$ **nonlinear:** As a first step, we consider

$$I_{u1}(t)=\frac{\vartheta}{t}\int_{t_0}^{t}\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds \ \text{ and } \ I_{u2}(t)=\frac{\vartheta}{t}\int_{t_0}^{t}\|u_{2r}(t)-u_{2r}(t-s)\|^2\,ds\ .$$

Then

$$\begin{aligned}I_{u1}(t) &\le \frac{\vartheta}{t}\int_{t_0}^{t}\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds\\ &\le \frac{2\vartheta}{t}\int_{t_0}^{t}\|u_{1r}(t)\|^2+\|u_{1r}(t-s)\|^2\,ds\\ &\le \frac{4\vartheta}{(1-\bar{\theta})t}\int_0^t(E(s)+E(t-s))ds\\ &\le \frac{8\vartheta}{(1-\bar{\theta})t}\int_0^t E(s)ds\\ &\le \frac{8\vartheta}{(1-\bar{\theta})t}\int_0^t E(0)ds = \frac{8\vartheta}{(1-\bar{\theta})}E(0),\end{aligned}$$

since $E(t)$ is non-increasing, $E(t)\le E(0)$. This gives the uniform bound $I_{u1}(t)\le\dfrac{8\vartheta}{1-\bar{\theta}}E(0)<\infty.$

Similarly, we derive

$$I_{u2}(t)\le\frac{8\vartheta}{1-\bar{\theta}}E(0)<\infty.$$

Choosing $0<\vartheta<1$, so that

$$I_{ui}(t)<1,\ \text{ for } i\in\{1,2,\}, \tag{16}$$

this will allow us to use the convexity properties of Θ.

Furthermore, the functions $w_{u1}(t)$ and $w_{u2}(t)$ are defined as

$$w_{u1}(t)=-\int_{t_0}^{t}\theta'(s)\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds,$$

and

$$w_{u2}(t)=-\int_{t_0}^{t}\theta'(s)\,\|u_{2r}(t)-u_{2r}(t-s)\|^2\,ds,$$

so

$$w_{ui}(t)\le -mE'(t),\quad i\in\{1,2\}.$$

As Θ is strictly convex on $(0,l]$ with $\Theta(0)=0$, then $\Theta(as)\le a\Theta(as)$ for $(a,s)\in(0,1)\times(0,l)$. By using (16), (**H2**) and Jensen's inequality, we arrive at

$$
\begin{aligned}
w_{u1}(t) &= \frac{1}{\vartheta I_{u1}(t)}\int_{t_0}^{t} I_{u1}(t)\left(-\theta'(s)\,\vartheta\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\right)ds \\
&\geq \frac{1}{\vartheta I_{u1}(t)}\int_{t_0}^{t} I_{u1}(t)\zeta(s)\Theta(\theta(s))\,\vartheta\,\|u_{1r}(t)-u_{1r}(t-s)\|^2 ds \\
&\geq \frac{\zeta(t)}{\vartheta I_{u1}(t)}\int_{t_0}^{t} \Theta\left(I_{u1}(t)\theta(s)\,\vartheta\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\right)ds \\
&\geq \frac{\zeta(t)}{\vartheta}\Theta\left(\vartheta\int_{t_0}^{t}\theta(s)\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds\right) \\
&\geq \frac{\zeta(t)}{\vartheta}\bar{\Theta}\left(\vartheta\int_{t_0}^{t}\theta(s)\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds\right)
\end{aligned}
$$

where, we extend $\bar{\Theta}$ to Θ so that $\bar{\Theta}\in C^2(0,\infty)$ that is strictly convex and strictly increasing. As a result,

$$
\int_{t_0}^{t}\theta(s)\,\|u_{1r}(t)-u_{1r}(t-s)\|^2\,ds \leq \frac{1}{\vartheta}\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u1}(t)}{\zeta(t)}\right).
$$

Similarly, we have

$$
\int_{t_0}^{t}\theta(s)\,\|u_{2r}(t)-u_{2r}(t-s)\|^2\,ds \leq \frac{1}{\vartheta}\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u2}(t)}{\zeta(t)}\right).
$$

Therefore, for $t\geq t_0$, (15) becomes

$$
\mathcal{L}'(t)\leq -kE(t)+k_3\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u1}(t)}{\zeta(t)}\right)+k_3\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u2}(t)}{\zeta(t)}\right). \tag{17}
$$

For $\epsilon_0 < l$, we define

$$
\mathcal{L}'(t)=E(t)+\bar{\Theta}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}(t),
$$

which is equivalent to $E(t)$.

From (17) and, $E'(t)\leq 0,\ \bar{\Theta}'>0$ and $\bar{\Theta}''>0$, we get

$$
\begin{aligned}
\mathcal{L}'_1(t) &= {}_0\frac{E'(t)}{E(0)}\bar{\Theta}''\left({}_0\frac{E(t)}{E(0)}\right)\mathcal{L}(t)+\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right)\ \ '(t)+E'(t) \\
&\leq -kE(t)\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right)+k_2\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u1}(t)}{\zeta(t)}\right)\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right)+k_2\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u2}(t)}{\zeta(t)}\right)\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right).
\end{aligned} \tag{18}
$$

We define $\bar{\Theta}^*$ as the convex conjugate of Θ in the sense of Young. Hence,

$$
\bar{\Theta}^*(s)=s(\bar{\Theta}')^{-1}(s)-\bar{\Theta}\left((\bar{\Theta}')^{-1}(s)\right), \tag{19}
$$

by Young's inequality,

$$
XY_i\leq\bar{\Theta}^*(X)+\bar{\Theta}(Y_i),\quad i=1,2, \tag{20}
$$

with and $X=\bar{\Theta}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)$, $Y_1=\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u1}(t)}{\zeta(t)}\right)$ and $Y=\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u2}(t)}{\zeta(t)}\right)$

Using (4) and (18)-(20), we obtain

$$
\begin{aligned}
\mathcal{L}'(t) &= -kE(t)\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right)+k_3\bar{\Theta}^*\left(\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right)\right)+k_3\bar{\Theta}\left(\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u1}(t)}{\zeta(t)}\right)\right) \\
&\quad +k_3\bar{\Theta}^*\left(\bar{\Theta}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\right)+k_3\bar{\Theta}\left(\bar{\Theta}^{-1}\left(\frac{\vartheta w_{u1}(t)}{\zeta(t)}\right)\right) \\
&\leq -kE(t)\bar{\Theta}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)+2k_3\,{}_0\frac{E(t)}{E(0)}\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right)-2\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right)+k_3\,\vartheta\left(\frac{w_{u1}(t)}{\zeta(t)}+\frac{w_{u2}(t)}{\zeta(t)}\right) \\
&\leq -kE(t)\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right)+k_4\,{}_0\frac{E(t)}{E(0)}\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right)+k_4\,\vartheta\left(\frac{w_{u1}(t)}{\zeta(t)}+\frac{w_{u2}(t)}{\zeta(t)}\right).
\end{aligned}
$$

Multiplying by $\zeta(t)$ and use $\epsilon_0 \dfrac{E(t)}{E(0)} < l,\ \ \bar{\Theta}'\left({}_0\dfrac{E(t)}{E(0)}\right) = \Theta'\left({}_0\dfrac{E(t)}{E(0)}\right)$ and $wi(t) \le -mE'(t)$ for $i \in \{u_1, u_2\}$ then

$$\zeta(t)\mathcal{L}'_1(t) \le -kE(t)\zeta(t)\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right) + k_5\,{}_0\frac{E(t)}{E(0)}\zeta(t)\bar{\Theta}'\left({}_0\frac{E(t)}{E(0)}\right) - k_5E'(t).$$

Letting $\mathcal{L}_2 = \zeta\mathcal{L}_1 + CE.$ Then, for some $\lambda_1, \lambda_2 > 0$:

$$\lambda_1\mathcal{L}_2(t) \le E(t) \le \lambda_2\mathcal{L}_2(t), \tag{21}$$

for $n_1 > 0$, a constant and for all $t \ge t_0$

$$\mathcal{L}'_2(t) \le -n_1\zeta(t)\frac{E(t)}{E(0)}\Theta'\left(\frac{\epsilon_0 E(t)}{E(0)}\right) = -n_1\zeta(t)\Theta_2\left(\frac{E(t)}{E(0)}\right), \tag{22}$$

where $\Theta_2(t) = t\Theta'(\epsilon_0 t)$. Since $\Theta_2'(t) = \Theta'({}_0t) + {}_0t\Theta''({}_0t)$ by using Θ is strictly convex on $(0,l]$, we observe that $\Theta_2'(t),\ \Theta_2(t) > 0$, on $(0,l]$. Hence, with

$$G(t) = \lambda\frac{\mathcal{L}_2(t)}{E(0)},$$

by (21) and (22), we find

$$G(t) \sim E(t), \tag{23}$$

for $n_2 > 0$,

$$G'(t) \le -n_2\zeta(t)\Theta_2(G(t)) \quad \forall t > t_0.$$

Then, integrating over (t_0, t) yields

$$\int_{t_0}^{t} -\frac{G'(s)}{\Theta_2(G(s))}ds \ge n_2\int_{t_0}^{t}\zeta(s)ds \Rightarrow \frac{1}{\epsilon_0}\int_{\epsilon_0 G(t)}^{\epsilon_0 G(t_0)}\frac{1}{s\Theta'(s)}ds \ge n_2\int_{t_0}^{t}\zeta(s)ds$$

$$\Rightarrow G(t) \le \frac{1}{\epsilon_0}\Theta^{-1}\left(n_1\int_{t_0}^{t}\zeta(s)ds\right), \tag{24}$$

with $\Theta_1(t) = \int_t^l \frac{1}{s\Theta'(s)}ds$ which is strictly decreasing on $(0,l]$ and $\lim_{t\to 0}\Theta_1(t) = +\infty$.

Combining (23) and (24), we get (14).

Remark 11. *The previous results can be extended in several directions.*

We may consider system (1) with nonlinear coupling terms $f_1(u_1,u_2)$ and $f_2(u_1,u_2)$ instead of the linear terms bu_1 and bu_2 respectively. Under suitable assumptions on the nonlinearities, and by applying the same approach developed in this work, it is possible to establish the well-posedness and derive general decay rates for the resulting nonlinear system.

The model can also be extended by replacing the viscoelastic damping with a fractional viscous damping term, there by preserving the nonlocal character of the system. By combining the techniques used in this paper with fractional calculus tools, one can prove the well-posedness and obtain decay estimates for the corresponding fractional system.

5. Conclusions

This The study of the general stability of coupled viscoelastic wave systems remains a significant and challenging topic due to the weak and nonlocal nature of memory-type dissipation. This work analyzed a coupled wave system incorporating viscoelastic damping and established the global existence of solutions through the Faedo–Galerkin method. Under a wide range of the relaxation functions and by applying a Lyapunov method together with convexity arguments, we derived a general decay result. The resulting decay is not restricted to classical exponential or polynomial rates but instead depends explicitly on the properties of the relaxation kernel.

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The authors declare no conflict of interest.

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