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Article

Double-measure Pseudo- $\mathcal{S}$ -asymptotically ω -periodic Functions and Applications

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Abstract

The aim of this work is to present some new contributions on double measure $\mathcal{S}$ -asymptotically ω -periodic functions. First we give the central result of the paper which characterizes the translation invariance of the double-measure ergodic spaces. After, we establish many interesting results on the functional spaces, such as composition theorems, Volterra operator. The theory developed in this work generalizes classical results on asymptotically ω-periodic functions. Then we apply the obtained results to investigate the existence of double measure pseudo $\mathcal{S}$ -asymptotically ω-periodic mild solutions for some nonautonomous partial evolution equations.

Keywords

Translation invariance, Convolution operator, Ergodicity, Nonautonomous evolution equation, Evolution family

MSC

34C27; 34K14; 34K30; 35B15; 58D25; 65J08.

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1. Introduction

In the qualitative theory of differential equations, a fundamental question is to determine whether the asymptotic behavior of the solutions reflects the recurrent or asymptotic structure of the coefficients and external forcing terms. In this context, periodic, almost periodic, asymptotically periodic, and related classes of recurrent functions have proved to be powerful tools for describing long-time dynamics and establishing the existence of solutions with prescribed asymptotic behavior.

Integral transforms and special functions provide useful representations of solutions to fractional differential equations. Ata and Kiymaz introduced a generalized Mellin transform [1] and developed a generalized Laplace transform with applications to the Bagley-Torvik problem and harmonic vibration models [2]. Ata, Kıymaz, and Başkönüş investigated an RC circuit model involving a generalized Caputo derivative [3]. Other studies have developed special functions based on the Wright function [4], a generalized kernel [5], or two generalized M-series [6]. The M-Srivastava [7] and M-Lauricella [8] extensions further contribute to this framework through integral representations and applications to fractional differential equations.

The study of asymptotic periodicity raises a complementary question concerning the long-time behavior of solutions. In this direction, Cuevas and de Souza [9] investigated $\mathcal{S}$-asymptotically ω-periodic solutions of semilinear fractional integro-differential equations. Our framework replaces the pointwise control of the period increment by a two-measure averaged control: μ is used in the integration, whereas ν appears in the normalization. Under assumption (**H1**), **Theorem 11** establishes the inclusion of the class of $\mathcal{S}$-asymptotically ω-periodic functions into the more general class considered here. The composition properties and the Volterra operator are then used to establish the existence and uniqueness of mild solutions for the non-autonomous first-order problem studied in this paper. This connection should not, however, be interpreted as a direct extension of the fractional models mentioned above; rather, the present framework provides a broader two-measure setting in which the asymptotic behavior of solutions can be analyzed.

The study of asymptotic almost periodic goes back to Fréchet [10]. Henriquez et al. [11,12] developed the theory of -asymptotically -periodic functions and its applications to differential equations. Ait Dads and Lhachimi [13] subsequently introduced a measure-based pseudo-asymptotic framework and established its basic structural properties. These classes descried recurrent behavior evolution equations, neutral equations, and related asymptotic propertie [9,14,15,16].

Measure-based ergodicity makes it possible to separate the recurrent part of a function from a perturbation that is negligible in an averaged sense. The onemeasure framework of Ait Dads and Lhachimi [13] and the translation-invariance criteria of Gu and Li [17] provide the background for the present study. Here, two positive measures, μ and ν, play distinct roles: μ determines the integral, whereas ν determines its normalization. We call the resulting class doublemeasure pseudo $\mathcal{S}$-asymptotically ω-periodic. This separation allows the two measures to have different growth rates.

The introduction of this two-measures setting leads to a larger and richer family of weighted ergodic functions and requires a careful analysis of its structural properties. In particular, we investigate the invariance of the resulting spaces under translations, composition operators, and convolution operators. We also establish inclusion relations between the corresponding function classes and characterize their translation invariance under suitable assumptions on the measures. These results extend several known properties from the one-measure framework and provide the fundamental analytic tools needed for applications to nonautonomous differential equations.

To explain the role of the two measures more precisely, let μ and ν be positive Borel measures satisfying appropriate finiteness and growth assumptions. In the corresponding ergodic mean, μ determines the measure with respect to which the function is integrated, while ν determines the normalization factor. In particular, for a suitable function f, the ergodic component is characterized through a condition of the form.

$$\lim_{r\to\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\| f(t)\| \, d\mu(t)=0.$$

This formulation clearly illustrates the independent roles of μ and ν. When $\mu=\nu$, the above definition reduces to the classical one-measure framework. Hence, the double-measure construction contains the previously studied setting as a particular case.

Translation invariance is a central issue in the theory of almost periodic and pseudo almost periodic function spaces. Diagana et al. [18] developed a two-measure framework for pseudo almost periodic and pseudo almost automorphic solutions of evolution equations. Ait Dads and Lhachimi [13] established translation-invariance and composition properties in a one-measure pseudo-asymptotic setting. In the present work, we characterize translation invariance for the double-measure ergodic class through the following condition:

$$\textbf{(HP)} \forall\, \tau \in \mathbb{J},\ \forall A \subset \text{a Borel set},\ (P_r(A+\tau) \underset{r\to\infty}{\to} 0) \Leftrightarrow (P_r(A) \underset{r\to\infty}{\to} 0).$$

This condition provides a suitable measure-theoretic framework for recovering translation invariance without imposing unnecessarily restrictive assumptions on the measures.

We also study the relationship between the newly introduced spaces and the classical spaces of $\mathcal{S}$-asymptotically ω-periodic functions. More precisely, under suitable assumptions, we established inclusion results of the form

$$\mathcal{E}(\mathbb{J}, X; \mu, \nu) \subset CB(\mathbb{J}, X),$$

and characterize the situation in which these inclusions become equalities. These results are complemented by composition and convolution theorems, which are essential for treating nonlinear integral and differential equations.

The functional framework developed in this paper is then applied to nonautonomous differential equations in Banach spaces. Let X be a Banach space, let $A(t)$ be a family of linear operators generating an evolution family $\{U(t,s)\}_{t\geq s}$, and consider the problem

$$\left.\begin{cases} x'(t) = A(t)x(t) + f(t, x(t)) \text{ for } t \geq 0 \\ x(0) = x0. \end{cases}\right\} \tag{1}$$

Under appropriate assumptions on the evolution family and the nonlinear term, we investigate the existence and uniqueness of double-measure pseudo S-asymptotically ω-periodic mild solutions of (1).

The main tool is a fixed-point argument based on the Banach contraction principle, combined with the properties of the function spaces established in the preceding sections.

The main contributions of this paper can be summarized as follows:

(1) We introduce and systematically study the class of double-measure pseudo S-asymptotically ω-periodic functions in Banach spaces.

(2) We develop a new class of measure-ergodic functions involving two positive measures and clarify the distinct roles played by the integration and normalization measures.

(3) We establish several structural properties of the new functions spaces, including relations, composition results, and stability under suitable convolution operators.

(4) We characterize the translation invariance of the double-measure ergodic spaces under a measure-theoretic condition weaker than some of the assumptions commonly used in the one-measure setting.

(5) We apply the abstract framework to nonautonomous differential equations in Banach spaces and establish the existence and uniqueness of double-measure pseudo S-asymptotically ω-periodic mild solutions by means of the evolution family and the Banach contraction principle.

The remainder of the paper is organized as follows. In Section 2, we recall the basic notions and preliminary results concerning evolution families and the function spaces involved in the analysis. Section 3 is devoted to the introduction and fundamental properties of double-measure pseudo S-asymptotically ω-periodic functions and their associated ergodic spaces. In section 4, we establish the main structural properties of these spaces, including translation invariance, composition and convolution results. In Section 5, we apply the abstract results to the nonautonomous differential equation (1) and prove the existence and uniqueness of double measure pseudo S-asymptotically ω-periodic mild solutions. Finally, an illustrative example is presented to demonstrate the applicability of the abstract theory.

Table 1. List of notations of spaces used in the paper.

The Banach space of bounded continuous functions from $\mathbb{J}$ into X	$CB(\mathbb{J};X)$
The space of bounded continuous (μ,ν)-ergodic functions	$\mathcal{E}(\mathbb{J};X;\mu,\nu)$
The space of measurable (μ,ν)-ergodic functions.	$\mathcal{E}_e(\mathbb{J};X;\mu,\nu)$
The space of measurable functions having a bounded normalized (μ,ν)-mean.	$B(\mathbb{J};X;\mu,\nu)$
The space of S-asymptotically ω-periodic functions.	$SAP_\omega(\mathbb{J};X)$
The space of (μ,ν)-pseudo-$\mathcal{S}$-asymptotically ω-periodic functions.	$PSAP_\omega^{\mu,\nu}(\mathbb{J};X)$
The parameter-dependent $\mathcal{S}$-asymptotically ω-periodic class, uniformly on bounded subsets of X.	$SAP_{\omega,\mathrm{ub}}(\mathbb{J}\times X;Y)$
The parameter-dependent (μ,ν)-pseudo-$\mathcal{S}$-asymptotically ω-periodic class, uniformly on bounded subsets of X.	$PSAP_{\omega,\mathrm{ub}}^{\mu,\nu}(\mathbb{J}\times X;Y)$
The normalized (μ,ν)-size of the Borel set A on $\mathbb{J}_r$.	$P_r(A)=\dfrac{\mu(A\bigcap\mathbb{J}_r)}{\nu(\mathbb{J}_r)}$

(**H1**) denotes the comparison condition between $\mu(\mathbb{J}_r)$ and $\nu(\mathbb{J}_r)$.

(**HP**) denotes the equivalence of asymptotic negligibility of a Borel set and its translates.

(**Hyb**) denotes the two-sided comparison condition for translated Borel sets.

($\mathbf{T}_v$) denotes the growth condition imposed on the normalizing measure ν under translations of $\mathbb{J}_r$.

2. Definitions and Preliminary Results

We briefly introduce some further notations and basic facts which will be used throughout the paper. Throughout this paper, we work on a domain $\mathbb{J}\in\{\mathbb{R}^+,\mathbb{R}\}$, and set $\mathbb{J}_r:=[-r,r]\cap\mathbb{J}$. Let $(X,\|.\|)$ denote a Banach space and $CB(\mathbb{J},X)$ denotes the Banach space of continuous and bounded functions from $\mathbb{J}$ to X, equipped with the supremum norm $\|f\|_\infty=\sup_{t\in\mathbb{J}}\|f(t)\|$.

Definition 1 *[19] We say that $A(t)$ generates an ω-periodic $(\omega>0)$ exponentially stable evolutionary process, $(U(t,s))t\geq s$ in X, that is, a two-parameter family of bounded linear operators that satisfies the following conditions:*

i) $U(t,t)=I$ for all $t\in\mathbb{R}$.

ii) $U(t,s)U(s,r)=U(t,r)$ for all $t\geq s\geq r$.

iii) The *map* $(t,s)\to U(t,s)x$ is continuous for every fixed $x\in X$.

iv) $U(t+\omega,s+\omega)=U(t,s)$ for all $t\geq s$ (ω-periodicity).

v) There exist $k>0$ and $\alpha>0$ such that $\|U(t,s)\|\leq ke^{-\alpha(t-s)}$ for $t\geq s$.

Definition 2 *A function $f\in CB(\mathbb{J},X)$ is said to be $\mathcal{S}$-asymptotically ω-periodic (where $\omega>0$), if $\lim_{|t|\to+\infty}f(t+\omega)-f(t)=0$. In this case, we say that ω is an asymptotic period of f. We denote by $\mathcal{S}AP_\omega(\mathbb{J};X)$ the subspace of $CB(\mathbb{J},X)$ formed by all the $\mathcal{S}$-asymptotically ω-periodic functions.*

Example 3 $f:t\mapsto\sin(t)+\sin\ln(1+t^2)$, $f\in\mathcal{S}AP_{2\pi}(\mathbb{R},\mathbb{R})$, In fact, $f(t+2\pi)-f(t)=\sin\ln(1+(t+2\pi)^2)-\sin\ln(1+t^2)$.

By the mean value theorem, there exists $\theta\in]t,t+2\pi[$ such that

$$\left|\sin\lin(1+(t+2\pi)^2)\text{-}\sin\ln(1+t^2)\right|=\left|2\pi\frac{2\theta}{1+\theta^2}\cos\ln(1+\theta^2)\right|\to 0.$$

Remark 4 *Every asymptotically periodic function is S-asymptotically ω-periodic function.*

In [20], Bochner introduced the concept of almost automorphy, which is a generalization of the almost periodicity.

Definition 5 *A function* $g \in C(\mathbb{R}, X)$ *is called almost automorphic if, for any sequence of real numbers* $\left(t_n'\right)_n$, *there exists a subsequence of* $\left(t_n'\right)_n$, *denoted* $\left(t_n\right)_n$, *such that, for each* $t \in \mathbb{R}$.

$$\lim_{m\to+\infty}\lim_{n\to+\infty} g(t+t_n-t_m) = g(t)$$

Denote by $\mathrm{AA}(\mathbb{R}, X)$ *the set of all such functions.*

Example 6 $f(t) = \sin\dfrac{1}{2-\sin t-\sin \pi t}$ *is an almost automorphic function, which is not almost periodic, because it is not uniformly continuous.*

3. New developments in the theory of double measure pseudo $\mathcal{S}$-asymptotically ω-periodic functions

In this part, we make extensive use of measure theory to introduce and study the class of double-measure $\mathcal{S}$-asymptotically ω-periodic functions. These new notions generalize all the different notions of weighted ergodicity recently introduced in the literature. In contrast with [13], here the idea consists of using two positive measures instead of one. Doing so will provide us with a larger and richer class of weighted ergodic spaces. Basic properties of these new functions will be studied, including their translation invariance and compositions properties. etc....

Denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures on $\mathcal{B}$ satisfying the following conditions:

$$\text{For } \mathbb{J} \in \{\mathbb{R}^+, \mathbb{R}\}, \text{ set } \mathbb{J}_r := [-r, r] \cap \mathbb{J}.$$

$$\mu(\mathbb{J}) = +\infty \text{ and } \mu([a,b]) < +\infty, \text{ for all } a, b \in \mathbb{J}(a < b).$$

Let us introduce the following hypothesis:

(**H1**): For $\mu, \nu \in \mathcal{M}$, we have: $\limsup\limits_{r\to\infty} \dfrac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} < \infty$.

Definition 7 *For* $(\mu,\nu) \in \mathcal{M}$, *a function* $f \in CB(\mathbb{J}, X)$ *is called* (μ,ν)*-ergodic if*

$$\lim_{r\to\infty} \frac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \| f(t) \| \, d\mu(t) = 0.$$

We denote by $\mathcal{E}(\mathbb{J}, X, \mu, \nu)$ *the subspace of* $CB(\mathbb{J}, X)$ *formed by all the* (μ,ν)*-ergodic functions. We note that* $\mathcal{E}(\mathbb{J}, X, \mu) = \mathcal{E}(\mathbb{J}, X, \mu, \mu)$.

We extend the notion of ergodic functions to accommodate cases where the functions may be unbounded or discontinuous.

Definition 8 *[21] For* $\mu, \nu \in \mathcal{M}$, *define the space*

$$\mathcal{E}_e(\mathbb{J}, X; \mu, \nu) = \left\{ \begin{array}{l} f : \mathbb{J} \to X, \text{measurable with respect to the measure } \mu \\ \text{such that } \lim\limits_{r\to\infty} \dfrac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \| f(t) \| \, d\mu(t) = 0. \end{array} \right\}$$

For $\mu, \nu \in \mathcal{M}$, let

$$B(\mathbb{J}, X; \mu, \nu) = \left\{ \begin{array}{l} f : \mathbb{J} \to X, \text{measurable with respect to the measure } \mu, \text{ such that} \\ \lim\limits_{r\to\infty} \dfrac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \| f(t) \| \, d\mu(t) < \infty. \end{array} \right\}$$

Remark 9 *Let* $\mu, \nu \in \mathcal{M}$, *Then*

i) $\mathcal{E}_e(\mathbb{J}, X, \mu, \nu) \subset B(\mathbb{J}, X, \mu, \nu)$.

If, moreover (μ,ν), *satisfies* (***H1***), *then for* $1 \le p \le \infty$,

ii) $L^p(\mathbb{J}, X, \mu) \subset B(\mathbb{J}, X, \mu, \nu)$. (by Hölder's inequality)

Definition 10 *Let* $\omega > 0$ *and* $\mu, \nu \in \mathcal{M}$, *a function* $f \in CB(\mathbb{J}, X)$ *is called* (μ,ν)*-pseudo* $\mathcal{S}$*-asymptotically* ω*-periodic, if*

$$\lim_{r\to\infty}\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\| f(t+\omega)-f(t)\|\,d\mu(t)=0.$$

We denote by $\mathcal{PSAP}^{\mu,\nu}_{\omega,\mathrm{ub}}(\mathbb{J},X)$ *the subspace of* $CB(\mathbb{J},X)$ *formed by all the* (μ,ν) *-pseudo* $\mathcal{S}$ *-asymptotically* ω *-periodic functions.*

$$\mathcal{PSAP}^{\mu,\nu}_{\omega}(\mathbb{J},X)=\left\{f\in CB(\mathbb{J},X);\, f_\omega-f\in\mathcal{E}(\mathbb{J},X,\mu,\nu)\right\},\ \textit{where}\ f_\omega(t)=f(t+\omega)\ . \tag{2}$$

3.1 Main Results in $\mathcal{PSAP}^{\mu,\nu}_{\omega}(\mathbb{J},X)$

Let's define the bounded linear difference operator

$$\Delta_\omega: CB(\mathbb{J},X)\to CB(\mathbb{J},X)\ \text{ by }\ (\Delta_\omega f)(t):=f(t+\omega)-f(t)\ .$$

and define $C_0(\mathbb{J};X)=\left\{f\in C(\mathbb{J},X)\text{ such that }\lim_{|x|\to\infty}f(x)=0\right\}$.

Let us give the first essential result in this section.

Theorem 11 Let $\omega>0$ *The following statements are equivalent:*

(i) ***(H1)*** *holds.*

(ii) $PSAP^{\mu,\nu}_{\omega}(\mathbb{J},X),\ \|\cdot\|_\infty$ is a Banach space.

(iii) $C_0(\mathbb{J},X)\subset PSAP^{\mu,\nu}_{\omega}(\mathbb{J},X)$.

(iv) $C_0(\mathbb{J},X)\subset\mathcal{E}(\mathbb{J};X;\mu,\nu)$.

(v) $SAP_\omega(\mathbb{J},X)\subset PSAP^{\mu,\nu}_{\omega}(\mathbb{J},X)$.

For the proof, we need the following lemma:

Lemma 12 Let $f\in CB(\mathbb{J},X)$, and set $u_1(t)=\cos\left(\frac{\pi t}{\omega}\right)f(t)$ and $u_2(t)=\sin\left(\frac{\pi t}{\omega}\right)f(t)$. If $u_1,u_2\in PSAP^{\mu,\nu}_{\omega}(\mathbb{J},X)$, then $f_\omega+f\in\mathcal{E}(\mathbb{J},X;\mu,\nu)$.

proof. One has $u_1,u_2\in PSAP^{\mu,\nu}_{\omega}(\mathbb{J},X)$. Thus $\Delta_\omega u_1,\Delta_\omega u_2\in\mathcal{E}(\mathbb{J},X;\mu,\nu)$.

Using $\cos(\theta+\pi)=-\cos\theta$ and $\sin(\theta+\pi)=-\sin\theta$ we obtain

$$\Delta_\omega u_1(t)=u_1(t+\omega)-u_1(t)=-\cos\left(\frac{\pi t}{\omega}\right)\left(f(t+\omega)+f(t)\right),$$

$$\Delta_\omega u_2(t)=u_2(t+\omega)-u_2(t)=-\sin\left(\frac{\pi t}{\omega}\right)\left(f(t+\omega)+f(t)\right).$$

Hence

$$\|\Delta_\omega u_1(t)\|+\|\Delta_\omega u_2(t)\|=\left\|\left(f(t+\omega)+f(t)\right)\right\|\left(\left|\cos\left(\frac{\pi t}{\omega}\right)\right|+\left|\sin\left(\frac{\pi t}{\omega}\right)\right|\right)\geq\left\|\left(f(t+\omega)+f(t)\right)\right\|$$

because $|\cos\theta|+|\sin\theta|\geq 1$. Therefore, $f_\omega+f\in\mathcal{E}(\mathbb{J},X;\mu,\nu)$.

proof. of **Theorem** 11

(i) $\Rightarrow$ (ii). Assume (**H1**). We first show that $\mathcal{E}(\mathbb{J};X;\mu,\nu)$ is closed in $CB(\mathbb{J},X)$. Let (f_n) be a sequence in $\mathcal{E}(\mathbb{J},X;\mu,\nu)$ such that $f_n\to f$ uniformly on J. Choose $r_0>0$ such that $\nu(\mathbb{J}_r)>0$ for all $r\geq r_0$. By (**H1**), there exists $M>0$ such that $\frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)}\leq M$ for all $r\geq r_0$: For every $r\geq r_0$,

$$\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|f(t)\|\,d\mu(t)\leq\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|f(t)-f_n(t)\|\,d\mu(t)+\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|f_n(t)\|\,d\mu(t)$$

$$\leq M\|f-f_n\|_\infty+\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|f_n(t)\|\,d\mu(t)$$

Letting $r\to\infty$ gives

$$\limsup_{r\to\infty}\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|f(t)\|d\mu(t)\le M\|f-f_n\|_\infty .$$

Finally, letting $n\to\infty$, we obtain that $\limsup_{r\to\infty}\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|f(t)\|d\mu(t)=0$, hence $f\in\mathcal{E}(\mathbb{J},X;\mu,\nu)$. So $\mathcal{E}(\mathbb{J},X;\mu,\nu)$ is closed.

By definition,

$$PSAP_\omega^{\mu,\nu}(\mathbb{J},X)=\Delta_\omega^{-1}\left(\mathcal{E}(\mathbb{J},X;\mu,\nu)\right).$$

Since Δ_ω is continuous and $\mathcal{E}(\mathbb{J},X;\mu,\nu)$ is closed, its inverse image is closed. Hence $PSAP_\omega^{\mu,\nu}(\mathbb{J},X)$ is closed in $CB(\mathbb{J},X)$, so it is a Banach space with respect to the supremum norm.

(ii)→ (iii). Assume (ii), i.e. $PSAP_\omega^{\mu,\nu}(\mathbb{J},X)$ is closed in $CB(\mathbb{J},X)$. We claim that $C_c(\mathbb{J};X)\subset PSAP_\omega^{\mu,\nu}(\mathbb{J},X)$. Indeed, if $\varphi\in C_c(\mathbb{J},X)$ then $\Delta_\omega\varphi$ has compact support. Therefore $\int_{\mathbb{J}_r}\|\Delta_\omega\varphi(t)\|d\mu(t)$ is bounded independently of r. Since $\nu(\mathbb{J}_r)\underset{r\to\infty}{\to}\infty$ (because $\nu\in\mathcal{M}$), we get

$$\frac{1}{\nu(\mathcal{J}_r)}\int_{\mathcal{J}_r}\|\Delta_\omega\varphi(t)\|d\mu(t)\to 0\qquad(r\to\infty),$$

so $\Delta_\omega\varphi\in\mathcal{E}(\mathcal{J},X;\mu,\nu)$ and $\varphi\in PSAP_\omega^{\mu,\nu}(\mathbb{J},X)$.

Finally, $C_c(\mathbb{J},X)$ is dense in $C_0(\mathbb{J},X)$ with respect to $\|\cdot\|_\infty$. Since $PSAP_\omega^{\mu,\nu}(\mathbb{J},X)$ is closed and contains $C_c(\mathbb{J},X)$, we deduce

$$C_0(\mathbb{J},X)=\overline{C_0(\mathbb{J},X)}^{\|\cdot\|_\infty}\subset PSAP_\omega^{\mu,\nu}(\mathbb{J},X).$$

(iii)→ (iv). Assume $C_0(\mathbb{J},X)\subset PSAP_\omega^{\mu,\nu}(\mathbb{J},X)$. Let $g\in C_0(\mathbb{J},X)$, fix $x_0\in X$ with $\|x_0\|=1$ and set $f(t):=\|g(t)\|x_0$. Define

$$u_1(t):=\cos\left(\frac{\pi t}{\omega}\right)f(t),\qquad u_2(t):=\sin\left(\frac{\pi t}{\omega}\right)f(t).$$

Then $u_1,u_2\in C_0(\mathbb{J},X)\subset PSAP_\omega^{\mu,\nu}(\mathbb{J},X)$. Then, by Lemma (12) $f_\omega+f\in\mathcal{E}(\mathbb{J};X;\mu,\nu)$. Furthermore, using $\|x_0\|=1$ and the definition $f(t):=\|g(t)\|x_0$, we obtain

$$\|g(t)\|\le\|g(t+\omega)\|+\|g(t)\|=\|f(t+\omega)+f(t)\|$$

So $g\in\mathcal{E}(\mathbb{J};X;\mu,\nu)$, proving (iv).

(iv) → (i). We proceed by contradiction. Assume that (**H1**) is false, i.e.

$$\lim_{r\to\infty}\frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)}=\infty.$$

Then, there exists an increasing sequence $r\to\infty$ such that

$$\frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)}\ge n^2(n\ge 1).$$

Choose a continuous decreasing function α: $[0,\infty)\to(0,1]$ such that $\alpha(r_n)=\frac{1}{n}$ for all n. Let's put

$$\alpha(r)=\begin{cases}1\text{ for }0\le r\le r_1\\ \frac{1}{n}\frac{r_{n+1}-r}{r_{n+1}-r_n}+\frac{1}{n+1}\frac{r-r_n}{r_{n+1}-r_n}\text{ for }r\in[r_n,r_{n+1}]\end{cases}$$

The values at the endpoints coincide on adjacent intervals. So α is continuous and deceasing. Moreover $\frac{1}{n+1}\le\alpha(r)\le\frac{1}{n}(r_n\le r\le r_{n+1})$, which gives that $\alpha(r)\underset{r\to\infty}{\to}0$. Define $a(t):=\alpha(|t|)$. Then $a\in C_0(\mathbb{J})$ and for every $t\in\mathbb{J}_{rn}$ one has $a(t)\ge\alpha(r_n)=\frac{1}{n}$. Therefore,

$$\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_m}a(t)d\mu(t)\ge\frac{1}{\nu(\mathbb{J}_{rn})}\int_{\mathbb{J}_m}\frac{1}{n}d\mu(t)=\frac{1}{n}\frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)}\ge n.$$

So $a \notin \mathcal{E}(\mathbb{J};\mathbb{R};\mu,\nu)$. Taking $f(t) := a(t)x_0$ with $\|x_0\| = 1$, we have $f \in C_0(\mathbb{J},X)$ but $f \notin \mathcal{E}(\mathbb{J},X;\mu,\nu)$, contradicting (iv). Hence (**H1**) must hold. Hence statements (i)- (iv) are pairwise equivalent

(v)→ (iii) Assume (v) and let $f \in C_0(\mathbb{J},X)$. Since $C_0(\mathbb{J},X)$ is translation invariant, then $f_\omega - f \in C_0(\mathbb{J},X)$, consequently $f \in SAP_\omega(\mathbb{J},X) \subset \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{J},X)$. It follows that $C_0(\mathbb{J},X) \subset \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{J},X)$.

(iv)→ (v) Let $f \in SAP_\omega(\mathbb{J},X)$, then $f_\omega - f \in C_0(\mathbb{J};X) \subset \mathcal{E}(\mathbb{J};X;\mu,\nu)$, so $f \in \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{J},X)$, from which we obtain $SAP_\omega(\mathbb{J},X) \subset \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{J},X)$. This ends the proof.

In this part, we give a generalization of some results in [22].

Theorem 13 *The following assertions hold:*

1) $\mathcal{E}(\mathbb{J},X;\mu) \subset \mathcal{E}(\mathbb{J},X;\mu,\nu)$ *if and only if* $\limsup_{r\to+\infty} \frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} < +\infty.$

2) *If* $\liminf_{r\to+\infty} \frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} > 0$, *then* $\mathcal{E}(\mathbb{J},X;\mu,\nu) \subset \mathcal{E}(\mathbb{J},X;\mu).$

In particular,

If $0 < \liminf_{r\to+\infty} \frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} \le \limsup_{r\to+\infty} \frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} < +\infty,$ *then* $\mathcal{E}(\mathbb{J},X;\mu) = \mathcal{E}(\mathbb{J},X;\mu,\nu).$

3) $\lim_{r\to+\infty} \frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} = 0$ *if and only if* $\mathcal{E}(\mathbb{J},X;\mu,\nu) = CB(\mathbb{J},X).$

proof. The case $\mathbb{J} = \mathbb{R}$ results from **Theorem 3.1** in [22]. Suppose $\mathbb{J} = \mathbb{R}^+$. For every Borel set $A \subset \mathbb{R}$, the measures are extended by $\mu_e(A) = \mu(A \cap \mathbb{R}^+)$. Then μ_e and ν_e are positive Borel measures on $\mathbb{R}$ that are finite on compact sets and have infinite total mass. Now let $f \in CB(\mathbb{R}^+,X)$. Extended to a continuous and bounded function on $\mathbb{R}$ by

$$\begin{cases} \tilde{f}(t) = f(t), \ t \ge 0 \\ \tilde{f}(t) = f(0), \ t < 0 \end{cases}.$$

Then

$$\mu_e([-r,r]) = \mu([0,r]), \ \nu_e([-r,r]) = \nu([0,r]),$$

and

$$\int_{[-r,r]} \|\tilde{f}(t)\| \, d\mu_e(t) = \int_0^r \|f(t)\| \, d\mu(t) \text{ , Thus } f \in \mathcal{E}(\mathbb{R}^+;X;\mu,\nu) \Leftrightarrow \tilde{f} \in \mathcal{E}(\mathbb{R}^+;X;\mu_e,\nu_e).$$

By applying **Theorem 3.1** in [22] on $\mathbb{R}$ to the couple (μ_e, ν_e), we immediately obtain the three conclusions on $\mathbb{R}^+$.

3.2 Characterization of $\mathcal{E}(\mathbb{J},X,\mu,\nu)$

Theorem 14 *Let* $\mathbb{J} \in \{\mathbb{R},\mathbb{R}^+\}$ *and let* $\mu,\nu \in \mathcal{M}$ *satisfy (**H1**). Let* $I \subset \mathbb{J}$ *be a Borel set such that* $\mu(I) < +\infty$ *and* $\nu(I) < +\infty$ *; the case* $I = \varnothing$ *is allowed. If* $f \in CB(\mathbb{J},X)$ *, then the following assertions are equivalent:*

1) $f \in \mathcal{E}(\mathbb{J},X;\mu,\nu)$.

2) $$\lim_{r\to+\infty} \frac{1}{\nu(\mathbb{J}_r \setminus I)} \int_{\mathbb{J}_r \setminus I} \|f(t)\| d\mu(t) = 0.$$

3) For every $\varepsilon > 0$,

$$\lim_{r\to+\infty} \frac{\mu\left(\left\{t \in \mathbb{J}_r \setminus I : \|f(t)\| > \varepsilon\right\}\right)}{\nu(\mathbb{J}_r \setminus I)} = 0.$$

4) For every $\varepsilon > 0$, there exists a Borel set $I_\varepsilon \subset \mathbb{J}$, independent of r, such that $\mu(I_\varepsilon) < +\infty$, $\nu(I_\varepsilon) < +\infty$, and

$$\lim_{r\to+\infty} \frac{\mu\left(\left\{t \in \mathbb{J}_r \setminus I_\varepsilon : \|f(t)\| > \varepsilon\right\}\right)}{\nu(\mathbb{J}_r \setminus I_\varepsilon)} = 0.$$

proof. Since $\mathbb{J}_r$ increases to $\mathbb{J}$ and $\nu(\mathbb{J}) = +\infty$, one has $\nu(\mathbb{J}_r) \to +\infty$. If $B \subset \mathbb{J}$ is a Borel set and $\mu(B)$, $\mu(B) < +\infty$, then

$$\frac{\nu(\mathbb{J}_r \setminus B)}{\nu(\mathbb{J}_r)} = \frac{\nu(\mathbb{J}_r) - \nu(\mathbb{J}_r \cap B)}{\nu(\mathbb{J}_r)} = 1 - \frac{\nu(\mathbb{J}_r \cap B)}{\nu(\mathbb{J}_r)} \tag{3}$$

$0 \le \frac{\nu(\mathbb{J}_r \cap B)}{\nu(\mathbb{J}_r)} \le \frac{\nu(B)}{\nu(\mathbb{J}_r)}$ which tends to zero since $\nu(B) < \infty$ and $\nu(\mathbb{J}_r) \to \infty$.

Thus, the expression in (3) tends to 1. Moreover $\nu(\mathbb{J} \setminus B) = \nu(\mathbb{J}_r) - \nu(\mathbb{J}_r \cap B) \ge \nu(\mathbb{J}_r) - \nu(B)$ which tends to infinity. Thus, it is positive for sufficiently large r. In particular, all denominators below are positive for large r. Put $M = \|f\|_\infty$. Using these relations with $B = I$, we have

$$0 \le \frac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \|f(t)\| d\mu(t) - \frac{\nu(\mathbb{J}_r \setminus I)}{\nu(\mathbb{J}_r)} \frac{1}{\nu(\mathbb{J}_r \setminus I)} \int_{\mathbb{J}_r \setminus I} \|f(t)\| d\mu(t) \le \frac{M\mu(I)}{\nu(\mathbb{J}_r)}.$$

Therefore 1) and 2) are equivalent. For $\varepsilon > 0$, set $A_{r,\varepsilon} = \{t \in \mathbb{J}_r \setminus I : \|f(t)\| > \varepsilon\}$. Chebyshev's inequality gives

$$\varepsilon \frac{\mu(A_{r,\varepsilon})}{\nu(\mathbb{J}_r \setminus I)} \le \frac{1}{\nu(\mathbb{J}_r \setminus I)} \int_{\mathbb{J}_r \setminus I} \|f(t)\| d\mu(t),$$

so 2) implies 3).

Clearly, 3) implies 4) by taking $I_\varepsilon = I$. Assume 4). Fix $\varepsilon > 0$, choose I_ε as in 4), and put $A_{r,\varepsilon} = \{t \in \mathbb{J}_r \setminus I_\varepsilon : \|f(t)\| > \varepsilon\}$. Then

$$\frac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \|f(t)\| d\mu(t) \le M \frac{\mu(A_{r,\varepsilon})}{\nu(\mathbb{J}_r)} + \varepsilon \frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} + M \frac{\mu(I_\varepsilon)}{\nu(\mathbb{J}_r)}.$$

The first and third terms tend to zero. By (**H1**) there exists $C_0 > 0$ such that $\frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} \le C_0$, Consequently

$$\limsup_{r \to +\infty} \frac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \|f(t)\| d\mu(t) \le C_0 \varepsilon.$$

Since this holds for every $\varepsilon > 0$, the lim sup is zero; hence $f \in \mathcal{E}(\mathbb{J}; X; \mu, \nu)$. Therefore 4) implies 1), and the proof is complete.

Remark 15 Due to the relation (2), we can characterize the space $\mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{J}, X)$ in a manner similar to that used for $\mathcal{E}(\mathbb{J}, X, \mu, \nu)$.

4. Translation Invariance with Two Measures

Assume the comparability condition (**H1**). In particular, there exist $r_0 > 0$ and $C_0 > 0$ such that $\frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} \le C_0$ $(r \ge r_0)$. For a Borel set $A \subset \mathbb{J}$ and $r > 0$, define

$$P_r(A) = \frac{\mu(A \cap \mathbb{J}_r)}{\nu(\mathbb{J}_r)}.$$

For $\tau \in \mathbb{J}$ define $A + \tau = \{t + \tau : t \in A\} \subset \mathbb{J}$.

We will use the following standard approximation property.

Theorem 16 (***Regular approximation***). *Let $\mathbb{J} \in \{\mathbb{R}, \mathbb{R}^+\}$ and let $\mu \in \mathcal{M}$. For every Borel set $A \subset \mathbb{J}$ and every $\eta > 0$, there exist a set F closed in $\mathbb{J}$ and a set* U *open in $\mathbb{J}$ such that $F \subset A \subset U$ and $\mu(U \setminus F) < \eta$.*

This follows from the regularity of Borel measures that are finite on compact subsets of $\mathbb{J}$; see [[23], T**heorems 2.17(a)** and **2.18**].

4.1 Characterization of Translation Invariance

4.1.1 Equivalence between bi-invariance under translation and the condition (HP).

Recall: For any Borel set $A \subset \mathbb{J}$; we set

$$P_r(A)=\frac{\mu(A\cap\mathbb{J}_r)}{\nu(\mathbb{J}_r)}\,.$$

And we assume (**H1**):

Consider the following **two properties.**

(**BI**) **Bi-invariance under translation.** We say that $\mathcal{E}(\mathbb{J},X;\mu,\nu)$ is bi-invariant under translation if

$$\forall\tau\in\mathcal{J},\ \forall f\in CB(\mathcal{J},X),\ f\in\mathcal{E}(\mathcal{J},X;\mu,\nu)\Leftrightarrow T_\tau\in\mathcal{E}(\mathcal{J},X;\mu,\nu)\ .$$

(**HP**):

$$\forall\tau\in\mathbb{J},\ \forall f\subset\mathbb{J}\ \text{ a Borel set, }\left(P_r(A+\tau)\underset{r\to\infty}{\to}0\right)\Leftrightarrow\left(P_r(A)\underset{r\to\infty}{\to}0\right).$$

Remark 17 If $\mathbb{J}=\mathbb{R}$, *then we can replace "$\Leftrightarrow$" by "$\Rightarrow$" in (**BI**) and (**HP**).*

Let $f\in CB(\mathbb{J},X)$. For $\varepsilon>0$, we pose

$$A_{\varepsilon,f}=\left\{t\in\mathbb{J}:\left\|f(t)\geq\varepsilon\right\|\right\}.$$

By **Theorem 14**, with $I=\varnothing$, one has:

$$f\in\mathcal{E}(\mathbb{J},X;\mu,\nu)\Leftrightarrow\forall\varepsilon>0,\ P_r(A_{\varepsilon,f})\underset{r\to\infty}{\to}0$$

Now, we give the essential result in this subsection.

Assume the hypothesis

$$\textbf{(H1)}\ \limsup_{r\to\infty}\frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)}<+\infty.$$

The **Theorem 14**, applied with $I=\varnothing$, yields

$$f\in\mathcal{E}(\mathbb{J},X;\mu,\nu)\Leftrightarrow\forall\varepsilon>0,\ P_r(A_{\varepsilon,f})\to0. \tag{4}$$

Theorem 18 *Under (**H1**), the properties (**BI**) and (**HP**) are equivalent.*

proof.

1) (**HP**)$\to$(**BI**). Fix $\tau\subset\mathbb{J}$ and let $f\in CB(\mathbb{J},X)$. For all $x\in\mathbb{J}$, define

$$f_\tau(x)=T_\tau f(x)=f(x+\tau).$$

For all $\varepsilon>0$, one has

$$\tau+A_{\varepsilon,f\tau}=\tau+\left\{t\in\mathbb{J}:\left\|f(t+\tau)\right\|\geq\varepsilon\right\}=(\tau+\mathbb{J})\cap A_{\varepsilon,f}\ . \tag{5}$$

If $\mathbb{J}=\mathbb{R}$, then $\tau+\mathbb{J}=\mathbb{J}$ and $\tau+A_{\varepsilon,f\tau}=A_{\varepsilon,f}$.

If $\mathbb{J}=\mathbb{R}^+$, then $\tau+\mathbb{J}=[\tau,+\infty)$ and

$$(\tau+\mathbb{J})\cap A_{\varepsilon,f}=A_{\varepsilon,f}\setminus([0,\tau)\cap A_{\varepsilon,f})\ .$$

Therefore

$$0\leq P_r(A_{\varepsilon,f})-P_r\left((\tau+\mathbb{J})\cap A_{\varepsilon,f}\right)\leq\frac{\mu([0,\tau])}{\nu(\mathbb{J}_r)}\to0\ .$$

In both cases,

$$P_r(\tau+A_{\varepsilon,f_\tau})\to0\Leftrightarrow P_r(A_{\varepsilon,f})\to0\ .$$

By now applying (**HP**) to the Borel set A_{ε,f_τ}, we obtain

$$P_r(A_{\varepsilon,f_\tau})\to0\Leftrightarrow P_r(\tau+A_{\varepsilon,f_\tau})\to0\ .$$

By combining the two preceding equivalences, we have

$$P_r(A_{\varepsilon,f_\tau})\to0\Leftrightarrow P_r(A_{\varepsilon,f})\to0\ .$$

This equivalence holds for every $\varepsilon > 0$. Then the characterization (4) gives

$$T_\tau f \in \mathcal{E}(\mathbb{J}, X; \mu, \nu) \Leftrightarrow f \in \mathcal{E}(\mathbb{J}, X; \mu, \nu) \cdot$$

Thus, the property (**BI**) is proved.

2) (**BI**)→(**HP**) Let's fix $\tau \in \mathbb{J}$ and a Borel set $A \subset \mathbb{J}$. Thanks to Theorem 16, we can choose two open sets *U*, *V* and two closed sets *F*, *G* such that $F \subset A \subset U$, $\mu(U \setminus F) < 1$ and $G \subset A + \tau \subset V$, $\mu(V \setminus G) < 1$.

First implication: from $P_r(A) \to 0$ to $P_r(A + \tau) \to 0$.

First step: Construction of a neighborhood adapted to translation.

Consider

$$\tilde{U} = (U + \tau) \bigcup \big(\mathbb{J} \setminus (\mathbb{J} + \tau)\big) \cdot$$

If $\mathbb{J} = \mathbb{R}$, then $\mathbb{J} + \tau = \mathbb{J}$ and $\tilde{U} = U + \tau$.

If $\mathbb{J} = \mathbb{R}^+$, then

$$\mathbb{J} + \tau = [\tau, +\infty), \ \tilde{U} = (U + \tau) \bigcup [0, \tau) \cdot$$

In both case, $\mathbb{J} \setminus \tilde{U} = (\mathbb{J} \setminus U) + \tau$. The right-hand side is closed in $\mathbb{J}$. It follows that $\tilde{U}$ is open in $\mathbb{J}$.

For all $t \in \mathbb{J}$, $t + \tau$ belongs to $\mathbb{J} + \tau$. Therefore, it cannot belong to $\mathbb{J} \setminus (\mathbb{J} + \tau)$. Thus

$$t + \tau \in \tilde{U} \Leftrightarrow t + \tau \in U + \tau \Leftrightarrow t \in U \tag{6}$$

This equivalence is the precise reason for the introduction of $\tilde{U}$, particularly in the case $\mathbb{J} = \mathbb{R}^+$.

Second step: Urysohn's function and membership of $CB(\mathbb{J}, X)$.

As $G \subset A + \tau \subset U + \tau \subset \tilde{U}$, the closed sets G and $\mathbb{J} \setminus \tilde{U}$ are disjoint. Urysohn's lemma yields a function $k \in C(\mathbb{J}, [0,1])$ such that $k(t) = 1$ $(t \in G)$, $k(t) = 0$ $(t \in \mathbb{J} \setminus \tilde{U})$. Let's choose $x_0 \in X$ with $\|x_0\| = 1$ and consider $K(t) = k(t) x_0$. The function *K* is continuous. Moreover,

$$\|K(t)\| = k(t) \|x_0\| = k(t) \le 1 ,$$

thus, $K \in CB(\mathbb{J}, X)$. Its translation $T_\tau K$ is also continuous and bounded; hence $T_\tau K \in CB(\mathbb{J}, X)$. Thanks to (6), if $t \notin U$, then $t + \tau \notin \tilde{U}$, hence $k(t + \tau) = 0$. Therefore

$$\begin{aligned} &\frac{1}{\nu(\mathcal{J}_r)} \int_{\mathcal{J}_r} \|T_\tau K(t)\| d\mu(t) = \frac{1}{\nu(\mathcal{J}_r)} \int_{\mathcal{J}_r} k(t + \tau) d\mu(t) \\ &= \frac{1}{\nu(\mathcal{J}_r)} \int_{U \cap \mathcal{J}_r} k(t + \tau) d\mu(t) \le \frac{\mu(U \cap \mathcal{J}_r)}{\nu(\mathcal{J}_r)} \le P_r(A) + \frac{1}{\nu(\mathcal{J}_r)} \end{aligned} \cdot \tag{7}$$

The last inequality follows from $U \setminus A \subset U \setminus F$ and of $\mu(U \setminus F) < \eta$.

Now, assume that $P_r \to 0$. The right- hand side of (7) then tends to zero. By the definition of the ergodic space

$$T_\tau K \in \mathcal{E}(\mathbb{J}, X; \mu, \nu) \cdot$$

At this point, we use the reverse implication in (**BI**):

$$T_\tau K \in \mathcal{E}(\mathbb{J}, X; \mu, \nu) \Rightarrow K \in \mathcal{E}(\mathbb{J}, X; \mu, \nu) \cdot$$

Consequently

$$\frac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \|K(t)\| \, d\mu(t) \to 0 .$$

Third step: From $K \in \mathcal{E}(\mathbb{J}, X; \mu, \nu)$ to the asymptotic neglidibility of $A + \tau$.

As $k = 1$ on G,

$$\begin{aligned}&\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|K(t)\|\,d\mu(t)=\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}k(t)d\mu(t)\\&\geq\frac{\mu(G\cap\mathbb{J}_r)}{\nu(\mathbb{J}_r)}=P_r(G)\geq P_r(A+\tau)-\frac{1}{\nu(\mathbb{J}_r)}\end{aligned}\quad.\tag{8}$$

In fact, $(A+\tau)\setminus G\subset V\setminus G$ and $\mu(V\setminus G)<1$.

By rearranging (8), we obtain

$$0\leq P_r(A+\tau)\leq\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|K(t)\|\,d\mu(t)+\frac{1}{\nu(\mathbb{J}_r)}.$$

The two terms on the right-hand side tend to zero; thus $P_r(A+\tau)\to 0$. The logical chain of this first implication is as follows

$$P_r(A)\to 0\overset{(4)}{\Rightarrow}T_\tau K\in\mathcal{E}(\mathbb{J},X;\mu,\nu)\overset{(BI)}{\Rightarrow}K\in\mathcal{E}(\mathbb{J},X;\mu,\nu)\overset{(5)}{\Rightarrow}P_r(A+\tau)\to 0\ .$$

Second implication: from $P_r(A+\tau)\to 0$ to $P_r(A)\to 0$.

First step: Construction of the second Urysohn function.

One has $F+\tau\subset A+\tau\subset V$: The sets $F+\tau$ and $\mathbb{J}\setminus V$ are closed and disjoint in $\mathbb{J}$. The Urysohn's lemma guarantees the existence of a function $\ell\in C(\mathbb{J},[0,1])$ such that

$$\ell(t)=1\ (t\in F+\tau),\ \ell(t)=0\ (t\in\mathbb{J}\setminus V)\ .$$

Consider $L(t)=\ell(t)x_0$. The function L is continuous and

$$\|L(t)\|=\ell(t)\|x_0\|=\ell(t)\leq 1\ .$$

Consequently

$$L\in CB(\mathbb{J};X),\ T_\tau L\in CB(\mathbb{J};X)\ .$$

Since ℓ vanishes outside of V,

$$\begin{aligned}&\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|L(t)\|\,d\mu(t)=\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\ell(t)d\mu(t)\\&\geq\frac{\mu(V\cap\mathbb{J}_r)}{\nu(\mathbb{J}_r)}\leq P_r(A+\tau)+\frac{1}{\nu(\mathbb{J}_r)}\end{aligned}\quad.\tag{9}$$

Here, $V\setminus(A+\tau)\subset V\setminus G$ and $\mu(V\setminus G)<1$.

Suppose that $P_r(A+\tau)\to 0$. Estimation (9) implies that

$$L\in\mathcal{E}(\mathbb{J},X;\mu,\nu)\ .$$

This time, we use the forward implication in (**BI**):

$$L\in\mathcal{E}(\mathbb{J},X;\mu,\nu)\Rightarrow T_\tau L\in\mathcal{E}(\mathbb{J},X;\mu,\nu)\ .$$

Furthermore

$$\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|T_\tau L(t)\|\,d\mu(t)\to 0\ .$$

Second step: From $T_\tau L\in\mathcal{E}(\mathbb{J},X;\mu,\nu)$ to the asymptotic negligibility of A. If $t\in F$, then $t+\tau\in F+\tau$, hence $\ell(t+\tau)=1$. It follows that

$$\begin{aligned}&\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\|T_\tau L(t)\|\,d\mu(t)=\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\ell(t)d\mu(t)\\&\geq\frac{\mu(F\cap\mathbb{J}_r)}{\nu(\mathbb{J}_r)}=P_r(F)\geq P_r(A)-\frac{1}{\nu(\mathbb{J}_r)}\end{aligned}\quad.\tag{10}$$

Indeed, $A\setminus F\subset U\setminus F$ and $\mu(U\setminus F)<1$.

By rearranging (10), we obtain

$$0 \le P_r(A) \le \frac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \left\| T_\tau L(t) \right\| d\mu(t) + \frac{1}{\nu(\mathbb{J}_r)} .$$

The two terms on the right-hand side tend to zero; thus $P_r(A) \to 0$. The logical chain of this second implication is as follows:

$$P_r(A+\tau) \to 0 \overset{(9)}{\Rightarrow} L \in \mathcal{E}(\mathbb{J}, X; \mu, \nu) \overset{(BI)}{\Rightarrow} T_\tau L \in \mathcal{E}(\mathbb{J}, X; \mu, \nu) \overset{(10)}{\Rightarrow} P_r(A) \to 0 .$$

With both implications established for any Borel set $A \subset \mathbb{J}$ and all $\tau \in \mathbb{J}$, the property (**HP**) is proved.

Now, we introduce the following hypotheses to prepare for the central result of this work:

Hypothesis (**Hyb**). For all $\tau \in \mathbb{J}$, there exist a constant $\beta(\tau) > 0$ and a function $\varepsilon_\tau : [0,\infty[\to [0,\infty[$ such that $\varepsilon_\tau(r) \underset{r\to\infty}{\to} 0$ and for r large enough and every Borel set $B \subset \mathbb{J}_r$,

$$(\textbf{Hyb1}): \quad \mu(B+\tau) \le \beta(\tau)\mu(B) + \varepsilon_\tau(r)\nu(\mathbb{J}_r),$$

$$(\textbf{Hyb2}): \quad \mu(B) \le \beta(\tau)\mu(B+\tau) + \varepsilon_\tau(r)\nu(\mathbb{J}_r).$$

Hypothesis $(\mathbf{T}_\nu)$. For all $\tau \in \mathbb{J}$, there exist $C_\tau > 0$ and $r_\tau > 0$ such that

$$\frac{\nu(\mathbb{J}_r + |\tau|)}{\nu(\mathbb{J}_r)} \le C_\tau, \ r \ge r_\tau .$$

We give an example of measures (μ,ν) satisfying (**H1**), (T_ν) and (**Hyb**) when $\mathbb{J} = \mathbb{R}^+$.

We introduce the Borel set

$$H = \bigcup_{n\ge 1} \left(n^2, n^2+1\right) \subset \mathbb{R}^+ .$$

Definition of measures. Let $\nu = \lambda$, where λ is the Lebesgue measure on $\mathbb{R}^+$, and we define μ by

$$\mu(E) = \lambda(E \setminus H), \text{ for every Borel set } E \subset \mathbb{R}^+ .$$

Verification of (**H1**), $(\mathbf{T}_\nu)$ and (**Hyb**) with an unbounded term $o(\nu(\mathbb{J}_r))$.

1) $\mu \in \mathcal{M}$. Indeed for $[a,b] \subset \mathbb{R}^+$, we have $\mu([a,b]) \le b - a < +\infty$. Furthermore $\mu(\mathbb{R}^+) = +\infty$ because, for example,

$$[n^2+1,(n+1)^2] \subset \mathbb{R}^+ \setminus H, \ \lambda\left([n^2+1,(n+1)^2]\right) = 2n \ \text{ and } \sum_{n\ge 1} 2n \text{ diverges.}$$

2) (**H1**) for all $r > 0$, $\frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} \le 1$.

3) $(\mathbf{T}_\nu)$, for $\tau \ge 0$ and $r > \tau$, one has

$$\frac{\nu(\mathbb{J}_r + r)}{\nu(\mathbb{J}_r)} = \frac{r+\tau}{r} \le 2$$

4) **Verification of (Hyb)** with an unbounded $o(\nu(\mathbb{J}_r))$ term. Fix $\tau \ge 0$ and let $A \subset \mathbb{J}_r$ be a Borel set. We have

$$\mu(E) = \lambda(E) - \lambda(E \cap H), \ E \subset \mathbb{R}^+ \text{ a Borel set.}$$

As $\lambda(A+\tau) = \lambda(A)$, one has

$$\mu(A+\tau) = \lambda(A) - \lambda\left((A+\tau) \cap H\right), \ \mu(A) = \lambda(A) - \lambda(A \cap H) .$$

Thus

$$\mu(A+\tau) - \mu(A) = \lambda(A \cap H) - \lambda\left((A+\tau) \cap H\right) .$$

We deduce that

$$\begin{cases} \mu(A+\tau) \le \mu(A) + \lambda(A \cap H) \le \mu(A) + \lambda(H \cap [0,r]), \\ \mu(A) \le \mu(A+\tau) + \lambda\left((A+\tau) \cap H\right) \le \mu(A+\tau) + \lambda(H \cap [0,r]) \end{cases},$$

Since $\lambda(H \cap [0,r]) \le \lambda(H \cap [0,r+\tau])$, we obtain, with $\beta(\tau) = 1$,

$$\mu(A+\tau)\le\mu(A)+\rho_\tau(r)\ ,\ \ \mu(A)\le\mu(A+\tau)+\rho_\tau(r)\ ,$$

and

$$\rho_\tau(r)=\lambda(H\bigcap[0,r+\tau])\ .$$

$\rho_\tau(r)=o(\nu(\mathbb{J}_r))$: For $R>0$,

$$H\bigcap[0,R]\subset\bigcup_{1\le n^2<R}\left(n^2,\,n^2+1\right)$$

then

$$\lambda(H\bigcap[0,R])\le\lfloor\sqrt{R}\rfloor\le\sqrt{R}\ .$$

It follows that

$$\rho_\tau(r)\le\sqrt{r+\tau}\ .$$

As $\nu(\mathbb{J}_r)=r$, one has

$$\frac{\rho_\tau(r)}{\nu(\mathbb{J}_r)}\le\frac{\sqrt{r+\tau}}{r}\underset{r\to+\infty}{\to}0\ .$$

Then $\rho_\tau(r)=o(\nu(\mathbb{J}_r))$. Moreover $\rho_\tau(r)=\lambda(H\bigcap[0,r+\tau])\underset{r\to+\infty}{\to}\lambda(H)=+\infty$. In particular, it is unbounded. This proves (**Hyb**) with an unbounded error term.

Remark 19 ***(Necessity of an unbounded error term).*** *Let us fix* $\tau=\frac{1}{2}$ *and put* $A_r=H\bigcap[0,r]$. *Then* $\mu(A_r)=0$ *while*

$$A_r+\frac{1}{2}\supset\bigcup_{1\le n^2+1\le r}\left(n^2,\,n^2+1+\frac{1}{2}\right)=\bigcup_{1\le n\le\lfloor\sqrt{r-1}\rfloor}\left(n^2+\frac{1}{2},n^2+\frac{3}{2}\right)$$

then

$$\left(A_r+\frac{1}{2}\right)\setminus H\supset\bigcup_{1\le n\le\lfloor\sqrt{r-1}\rfloor}\left(n^2+1,n^2+\frac{3}{2}\right).$$

It follows that

$$\mu(A_r+\frac{1}{2})\ge\frac{1}{2}\lfloor\sqrt{r-1}\rfloor\ \textit{and}\ \ \mu(A_r)=0\ ,$$

therefore any inequality of the form $\mu(A+\tau)\le\beta(\tau)\mu(A)+\rho_\tau(r)$ *gives*

$$\mu(A_r+\frac{1}{2})\le\beta(\frac{1}{2})\mu(A_r)+\rho_{\frac{1}{2}}(r)\ ,$$

thus $\frac{1}{2}\lfloor\sqrt{r-1}\rfloor\le\rho_{\frac{1}{2}}(r)$ *which forces* $\rho_{\frac{1}{2}}(r)$ *to be unbounded. (Recall that* $\rho(r)=\varepsilon(r)\nu(\mathbb{J}_r)$

Lemma 20 *If* $\mu=\nu$ *and* ***(Hyb1)*** *holds, then, for all* $\tau\in\mathbb{J}$, *there exist* $C_\tau>0$ *and* $r_\tau>0$ *such that*

$$\frac{\mu(\mathbb{J}_r+|\,\tau\,|)}{\mu(\mathbb{J}_r)}\le C_\tau,\ r>r_\tau\ .$$

In other words, $(\mathbf{T}_\nu)$ *holds with* $\nu=\mu$.

proof. We fix $\tau\in\mathbb{J}$ and we apply (**Hyb1**) with $B=\mathbb{J}_r$:

$$\mu(\mathbb{J}_r+\tau)\le\beta(\tau)\mu(\mathbb{J}_r)+o(\mu(\mathbb{J}_r)),\ (r\to+\infty)\ .\tag{11}$$

If $\mathbb{J}=\mathbb{R}$; we also have $-\tau\in\mathbb{J}$ and then

$$\mu(\mathbb{J}_r-\tau)\le\beta(-\tau)\mu(\mathbb{J}_r)+o(\mu(\mathbb{J}_r))\ .\tag{12}$$

For $r\ge|\,\tau\,|$, one has

$$\mathbb{J}_r+|\,\tau\,|=(\mathbb{J}_r+\tau)\bigcup(\mathbb{J}_r-\tau)\ ,$$

hence, by subadditivity, and (11)-(12)

$$\mu(\mathbb{J}_r+|\tau|)\le\big(\beta(\tau)+\beta(-\tau)\big)\mu(\mathbb{J}_r)+o(\mu(\mathbb{J}_r))\ .$$

If $\mathbb{J}=\mathbb{R}^+$, one has $\tau\ge 0$ and

$$\mathbb{J}_{r+\tau}=(\mathbb{J}_r+\tau)\bigcup\mathbb{J}_r\ .$$

Thus by (11)

$$\mu(\mathbb{J}_{r+\tau})\le\beta(\tau)\mu(\mathbb{J}_r)+o\big(\mu(\mathbb{J}_r)\big)+\mu(\mathbb{J}_r)\ .$$

In both cases, we obtain

$$\mu(\mathbb{J}_{r+\tau})\le C_\tau\mu(\mathbb{J}_r)$$

for r large enough, which proves **Lemma 20**.

Proposition 21 Under the hypotheses (**Hyb**) and $(\mathbf{T}_\nu)$, the hypothesis (**HP**) is satisfied.

proof. Fix $\tau\in\mathbb{J}$ (Note that $\tau\ge 0$ when $\mathbb{J}=\mathbb{R}^+$). For $\tau>0$, set $s=r+|\tau|$.

Step 1: Inequality in the direction $A\mapsto A+\tau$. For $r>0$, we have the following set identity and inclusion

$$(A+\tau)\bigcap\mathbb{J}_r=(A\bigcap(\mathbb{J}_r-\tau))+\tau\subset(A\bigcap\mathbb{J}_s)+\tau\ .$$

Then

$$\mu\big((A+\tau)\bigcap\mathbb{J}_r\big)\le\beta(\tau)\mu(A\bigcap\mathbb{J}_s)+o(\nu(\mathbb{J}_s))\ .$$

By dividing by $\nu(\mathbb{J}_s)$ and using $(\mathbf{T}_\nu)$, we obtain

$$P_r(A+\tau)\le\frac{\nu(\mathbb{J}_s)}{\nu(\mathbb{J}_r)}[\beta(\tau)P_s(A)+o(1)]\le\beta(\tau)C_\tau P_s(A)+o(1)\ . \tag{13}$$

Step 2: the inequality in the opposite direction. Letís put $A_r:=A\bigcap\mathbb{J}_r\subset\mathbb{J}_r$. By the second inequality of (**Hyb**),

$$\mu(A_r)\le\beta(\tau)\mu(A_r+\tau)+o(\nu(\mathbb{J}_r))\ .$$

Since $A_r+\tau\subset(A+\tau)\bigcap\mathbb{J}_s$, we have

$$\mu(A_r+\tau)\le\mu\big((A_r+\tau)\bigcap\mathbb{J}_r\big),$$

Hence

$$\mu(A_r)\le\beta(\tau)\mu\big((A_r+\tau)\bigcap\mathbb{J}_s\big)+o(\nu(\mathbb{J}_r))\ .$$

By dividing by $\nu(\mathbb{J}_r)$ and using $(\mathbf{T}_\nu)$, we obtain

$$P_r(A)\le\beta(\tau)\frac{\nu(\mathbb{J}_s)}{\nu(\mathbb{J}_r)}P_s(A+\tau)+o(1)\le\beta(\tau)C_\tau P_s(A+\tau)+o(1)\ . \tag{14}$$

Step 3: Conclusion (HP). If $P_r(A)\to 0$, then $P_s(A)=P_{r+|\tau|}(A)\underset{r\to\infty}{\to}0$ and (13) leads to $P_r(A+\tau)\underset{r\to\infty}{\to}0$. Conversely, if $P_r(A+\tau)\to 0$, then $P_s(A+\tau)=P_{r+|\tau|}(A+\tau)\underset{r\to\infty}{\to}0$ and (14) implies that $P_r(A)\underset{r\to\infty}{\to}0$, which proves (**HP**).

Remark 22 If $\mathbb{J}=\mathbb{R}$, $\mu=\nu$, and (**Hyb1**) holds, then (**HP**) is satisfied. Indeed from Lemma 20, $(\mathbf{T}_\nu)$ holds, so thanks to **step 1** of **Proposition 21** above $P_r(A)\to 0$ implies $P_r(A+\tau)\to 0$, and the fact that $\mathbb{J}=\mathbb{R}$, leads to (**HP**).

Corollary 23. *Suppose that $\mu=\nu$. If (**Hyb**) is satisfied, then (**HP**) holds.*

proof. Since (**Hyb**) includes (**Hyb1**), **Lemma 21** implies that the condition $(\mathbf{T}_\nu)$ is automatically satisfied. The result then follows directly from **Proposition 22**.

Remark 24: *We give an example of measures for which (**HP**) holds but $(\mathbf{T}_\nu)$ fails.*

$\mathbb{J}=\mathbb{R}^+$, $d\mu(t)=e^{t^2}dt$, $\nu=\mu$, $\mu_r=\mu([0,r]$, *We have*

$$\mu_r\underset{r\to\infty}{\sim}\int_1^r e^{t^2}dt\underset{r\to\infty}{\sim}\int_1^r(1-\frac{1}{2t^2})\,e^{t^2}dt=\int_1^r(\frac{e^{t^2}}{2t})'dt\underset{r\to\infty}{\sim}\frac{e^{r^2}}{2r} \tag{15}$$

For every fixed $\tau > 0$, *we have*

$$\frac{\nu([0,r+\tau])}{\nu[0,r]} \underset{r\to\infty}{\sim} \frac{r}{r+\tau} e^{2r\tau+\tau^2} \to +\infty ,$$

and for $r > \tau$,

$$\frac{\mu([0,r-\tau])}{\mu[0,r]} \sim \frac{r}{r+\tau} e^{-2r\tau+\tau^2} \underset{r\to\infty}{\to} 0 .$$

The first limit proves that $(\mathbf{T}_v)$ *fails. Let* A *be a Borel set of* $\mathbb{R}^+$. *Then* $\mu(A \bigcap [0,r-\tau[) \le \mu_{r-\tau} = o(\mu_r)$. *It follows that*

$$P_r = \frac{\mu(A \bigcap [0,r])}{\mu_r} = \frac{\mu(A \bigcap [0,r-\tau[)}{\mu_r} + \frac{\mu(A \bigcap [r-\tau,r])}{\mu_r}$$
$$= o(1) + \frac{\mu(A \bigcap [r-\tau,r])}{\mu_r}$$

Hence

$$P_r(A) \to 0 \quad \textit{if and only if} \quad \frac{\mu(A \bigcap [r-\tau,r])}{\mu_r} \underset{r\to\infty}{\to} 0 . \tag{16}$$

On the one hand, we have

$$\mu(E+\tau) = \int_{E+\tau} e^{u^2} du = \int_E e^{(s+\tau)^2} ds = e^{\tau^2} \int_E e^{2\tau s} e^{s^2} ds$$

$$\forall E \subset [a,b] \Rightarrow e^{\tau^2+2a\tau} \mu(E) \le \mu(E+\tau) \le e^{\tau^2+2b\tau} \mu(E) . \tag{17}$$

On the other hand

$$\mu\big((A+\tau) \bigcap [r-\tau,r]\big) = \mu\big((A \bigcap [r-2\tau,r-\tau]+\tau)\big) ,$$

then, by (17) we deduce that, for $r > 2\tau$

$$(II) \le \frac{\mu\big((A+\tau) \bigcap [r-\tau,r]\big)}{\mu_r} \le (I) \tag{18}$$

where

$$(I) = \frac{e^{\tau^2+2(r-\tau)\tau}}{\mu_r} \mu\big(\big(A \bigcap [r-2\tau,r-\tau]\big)\big) \underset{r\to\infty}{\sim} \frac{\mu\big(\big(A \bigcap [r-2\tau,r-\tau]\big)\big)}{\mu_r - \tau} \tag{19}$$

and

$$(II) = \frac{e^{\tau^2+2(r-2\tau)\tau}}{\mu_r} \mu\big(\big(A \bigcap [r-2\tau,r-\tau]\big)\big) \underset{r\to\infty}{\sim} e^{-2\tau^2} \frac{\mu\big(\big(A \bigcap [r-2\tau,r-\tau]\big)\big)}{\mu_r - \tau} \tag{20}$$

from (18), (19) and (20), we deduce that

$$\frac{\mu\big(\big(A \bigcap [r-2\tau,r-\tau]\big)\big)}{\mu_r - \tau} \to 0 \ \textit{if and only if} \ \frac{\mu\big(\big(A+\tau) \bigcap [r-\tau,r]\big)\big)}{\mu_r} \to 0 ,$$

Thus, by (16) we have that: $P_r(A) \underset{r\to\infty}{\to} 0$ *if and only if* $P_r(A+\tau) \underset{r\to\infty}{\to} 0$. *Hence* (***HP***) *holds.*

4.1.2 Special case: Translation invariance of $\mathcal{E}(\mathbb{R}, X, \mu)$ $(\mathbb{J} = \mathbb{R}, \nu = \mu)$

This problem was considered for the first time by Blot et al. in [24]. Later, this setting was generalized by Ait Dads et al. in [13] under the hypothesis (**H0**). An other generalization was given in [17] under the hypothesis (**HG**).

It is well known that many important properties of measure-pseudo S - asymptotically ω -periodic functions are based on the translation invariance; see [13,16,17,18]. In this part, we introduce a new hypothesis and give a new result of the translation invariance with conditions weaker than the known ones.

Let μ be a positive Borel measure on R, finite on every compact set. For $r > 0$ set

$$I_r = [-r,r],\ P_r(A) = \frac{\mu(A \bigcap I_r)}{\mu(I_r)},\ (A \subset \mathbb{R} \text{ a Borel set}) .$$

Assume that $\mu(I_r) \underset{r\to+\infty}{\to} +\infty$.

In what follows, we consider the following hypothesis:

Hypothesis (**HG**) (Gu-Li type condition, one measure) introduced by Chuan- Yun Gu and Hong-Xu Li in [17].

Hypothesis (**HG**): For every $\tau \in \mathbb{R}$ there exist a constant $\beta > 0$ and a measurable set $\Omega \subset \mathbb{R}$ such that:

(**HG1**) for every Borel set $A \subset \mathbb{R}$ with $A \bigcap \Omega = \varnothing$, $\mu(A+\tau) \le \beta\mu(A)$.

(**HG2**) the exceptional set is negligible at the ergodic scale,

$$\frac{\mu\big((\Omega \bigcap I_r)+\tau\big)}{\mu(I_r)} \underset{r\to+\infty}{\to} 0.$$

Hypothesis (**H0**). For every $\tau \in \mathbb{R}$ there exists $\alpha_\tau > 0$ such that for every Borel set $A \subset \mathbb{R}$,

$$\mu(A+\tau) \le \alpha_\tau(1+\mu(A)).$$

Hypothesis (**H0**): was introduced by Ait Dads et al. in [13].

4.1.3 Relationships between hypotheses (H0), (HG), and (Hyb). in relation to (HP)

Proposition 25 *Assume that hypothesis (**HG**) satisfied. Then (**HP**) holds.*

proof. Fix $\tau \in \mathbb{R}$ and a Borel set $A \subset I_r$. Then,

$$\begin{aligned}\mu(A+\tau) &= \mu\big((A \bigcap \Omega^c)+\tau\big) + \mu\big((A \bigcap \Omega)+\tau\big)\\ &\le \beta\mu(A \bigcap \Omega^c) + \mu\big((I_r \bigcap \Omega)+\tau\big) \le \beta\mu(A) + o\big(\mu(I_r)\big).\end{aligned}$$

Hence (**Hyb1**) holds. By Remark 22, we conclude that (**HP**) is satisfied.

Proposition 26 Assume that hypothesis (**H0**) is satisfied. Then (**Hyb**) holds.

proof. (**Hyb1**) follows immediately from $\alpha_\tau = o\big(\mu(I_r)\big)$. To obtain (**Hyb2**) it suffices to apply (**H0**) to the set $A+\tau$ and to the translation $-\tau$.

Now, we introduce the hypothesis (**H2**): $\limsup_{r\to\infty} \frac{\mu(\mathbb{J}_r)}{\nu(\mathbb{J}_r)} > 0$.

Let us consider $\mathbf{AA}(\mathbb{R},\mathbf{X})$, the Banach space of almost automorphic functions from $\mathbb{R}$ to X. First, let's recall the following lemma:

Lemma ***27*** *[22] Let $\mu,\nu \in \mathcal{M}$ satisfy (**H2**) and assume that $\mathcal{E}(\mathbb{R},\mathbb{X};\mu,\nu)$ is translation invariant. If $f = g + h$ where $g \in \mathbf{AA}(\mathbb{R},\mathbf{X})$ and $h \in (\mathbb{R},\mathbb{X};\mu;\nu)$, then $g(\mathbb{R}) \subset \overline{f(\mathbb{R})}$.*

Proposition 28 *Under the hypotheses (**H1**), (**H2**), and (**HP**), the following assertions hold:*

i) The sum $\mathbf{AA}(\mathbb{R},\mathbf{X}) + \mathcal{E}(\mathbb{R},\mathbb{X};\mu,\nu)$ is direct.

ii) $[\mathbf{AA}(\mathbb{R},\mathbf{X}) \oplus \mathcal{E}(\mathbb{R},\mathbb{X};\mu,\nu)] \bigcap \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R},X) = P_\omega(\mathbb{R},X) \oplus \mathcal{E}(\mathbb{R},\mathbb{X};\mu,\nu)$.

where $P_\omega(\mathbb{R},X) = \{f : \mathbb{R} \to X$ continuous and ω-periodic$\}$.

proof. i) Let $g \in \mathbf{AA}(\mathbb{R},\mathbf{X}) \bigcap \mathcal{E}(\mathbb{R},\mathbb{X};\mu,\nu)$. Then $0 = g + (-g)$. By **Lemma 27**, it follows that $g(\mathbb{R}) \subset \{0\}$. Thus $g = 0$.

ii) Since $P_\omega(\mathbb{R},X) \subset \mathbf{AA}(\mathbb{R},\mathbf{X})$, $P_\omega(\mathbb{R},X) \subset \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R},X)$ and thanks to (**HP**), we have $\mathcal{E}(\mathbb{R},\mathbb{X};\mu;\nu) \subset \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R},X)$, so

$$P_\omega(\mathbb{R},X) \oplus \mathcal{E}(\mathbb{R},\mathbb{X};\mu;\nu) \subset [\mathbf{AA}(\mathbb{R},\mathbf{X}) \oplus \mathcal{E}(\mathbb{R},\mathbb{X};\mu;\nu)] \bigcap \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R},X).$$

Moreover, let $f \in [\mathbf{AA}(\mathbb{R},\mathbf{X}) \oplus \mathcal{E}(\mathbb{R},\mathbb{X};\mu;\nu)] \bigcap \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R},X)$, Then $f = g + \varphi$, with $g \in \mathbf{AA}(\mathbb{R},\mathbf{X})$ and $\varphi \in \mathcal{E}(\mathbb{R},\mathbb{X};\mu;\nu)$. We have $f_\omega - f \in \mathcal{E}(\mathbb{R},\mathbb{X};\mu;\nu)$. On the other hand, $f_\omega - f = (g_\omega - g) + (\varphi_\omega - \varphi)$, by the translation invariance, we have $(g_\omega - g) \in \mathbf{AA}(\mathbb{R},\mathbf{X})$ and $(\varphi_\omega - \varphi) \in \mathcal{E}(\mathbb{R},\mathbb{X};\mu;\nu)$. Since the sum is direct, $f_\omega - f = \varphi_\omega - \varphi$ and $0 = g_\omega - g$. Then $g \in P_\omega(\mathbb{R},X)$, which completes the proof.

4.1.4 The special case of Lebesgue measure

Hypothesis (**HP**) is automatic and the inclusion is strict

Let us fix $\mu=\lambda$ where λ denotes the Lebesgue measure on $\mathbb{R}^+$. Consider $\nu\in\mathcal{M}$, such that $\nu(\mathbb{R}^+)=+\infty$. We put $V(r)=\nu([0,r]),\ r>0$. The hypothesis (**H1**) becomes $\limsup_{r\to+\infty}\frac{r}{V(r)}<+\infty$.

Proposition 29 (*HP*) *is automatic when* $\mu=\lambda$.

proof. Let us fix a Borel set $A\subset\mathbb{R}^+$ and a number $\tau\geq 0$. For $r>\tau$, one has $(A+\tau)\bigcap[0,r]=\big(A\bigcap[0,r-\tau]\big)+\tau$. By translation invariance of the Lebesgue measure, $\lambda\big((A+\tau)\bigcap[0,r]\big)=\lambda\big(A\bigcap[0,r-\tau]\big)$. So $\lambda\big((A+\tau)\bigcap[0,r]\big)\leq\lambda\big(A\bigcap[0,r]\big)$. By dividing by $V(r)$, one obtains $P_r(A+\tau)\leq P_r(A)$. Thus $P_r(A)\underset{r\to\infty}{\to}0$ implies $P_r(A)\to 0$.

Conversely, for $r>\tau$, one has $A\bigcap[0,r]\subset\big(A\bigcap[0,r-\tau]\big)\bigcup[r-\tau,r]$. Thus $\lambda\big(A\bigcap[0,r]\big)\leq\lambda\big(A\bigcap[0,r-\tau]\big)+\tau$. But

$$\lambda\big(A\bigcap[0,r-\tau]\big)=\lambda\big((A+\tau)\bigcap[\tau,r]\big)\leq\lambda\big((A+\tau)\bigcap[0,r]\big).$$

Consequently, $\lambda\big(A\bigcap[0,r]\big)\leq\lambda\big((A+\tau)\bigcap[0,r]\big)+\tau$. By dividing by $V(r)$, one obtains $P_r(A)\leq P_r(A+\tau)+\frac{\tau}{V(r)}$. As $V(r)\underset{r\to\infty}{\to}+\infty$, one has $\frac{\tau}{V(r)}\to 0$. So $P_r(A+\tau)\underset{r\to\infty}{\to}0$ implies $P_r(A)\underset{r\to\infty}{\to}0$, which proves that (**HP**) holds.

Proposition 30 *Condition (**H1**) and* $\liminf_{r\to+\infty}\frac{r}{V(r)}=0$ *hold if and only if* $\mathcal{E}(\mathbb{R}^+,X,\lambda)\subsetneq\mathcal{E}(\mathbb{R}^+,X,\lambda,\nu)$.

proof. Sufficient condition: By (**H1**), we know that $\mathcal{E}(\mathbb{R}^+,X,\lambda)\subset\mathcal{E}(\mathbb{R}^+,X,\lambda,\nu)$.

Now, by hypothesis we have that $\liminf_{r\to+\infty}\frac{r}{V(r)}=0$, then we can therefore construct a sequence $(b_n)_n\geq 1$ by induction such that $b_n>2b_{n-1},\ n\geq 2,\ b_n>\sum_{k=1}^{n-1}b_k$, $n>2$, and $\frac{V(b_n)}{b_n}\geq n,\ n\geq 1$. Let us put $S_n=\sum_{k=1}^{n}b_k$. For $n\geq 2$, as $b_n>\sum_{k=1}^{n-1}b_k$, one has $S_n<2b_n$. Since $V(b_n)\geq nb_n$, we obtain $\frac{S_n}{V(b_n)}\leq\frac{2b_n}{nb_n}=\frac{2}{n}$.

Let's choose a function $\psi\in C\big([0,1];[0,1]\big)$ such that $\psi(0)=\psi(1)=0$ and $c_0:=\int_0^1\psi(s)ds>0$.

We define a scalar function $\varphi:\mathbb{R}^+\to[0,1]$ in the following way. For $t\in[b_n,2b_n]$, we put $\varphi(t)=\varphi\left(\frac{t-b_n}{b_n}\right)$. Outside the union of the intervals $[b_n,2b_n]$, we set up $\varphi(t)=0$. The intervals $[b_n,2b_n]$ are pairwise disjoint because $b_{n+1}>2b_n$. As $\psi(0)=\psi(1)=0$, the function φ is continuous. Moreover, $0\leq\varphi(t)\leq 1$, $t\geq 0$.

For all $n\geq 1$, one has $\int_{b_n}^{2b_n}\varphi(t)dt=b_n\int_0^1\psi(s)ds=c_0b_n$. We choose $x_0\in X$ with $\|x_0\|=1$. We set $f(t)=\varphi(t)x_0$. Then $f\in CB(\mathbb{R}^+;X)$ and $\|f(t)\|=\varphi(t)$. For $r=2b_n$, one has

$$\frac{1}{2b_n}\int_{b_n}^{2b_n}\|f(t)\|dt\geq\frac{1}{2b_n}\int_{b_n}^{2b_n}\varphi(t)dt.$$

So

$$\frac{1}{2b_n}\int_{0}^{2b_n}\|f(t)\|dt\geq\frac{c_0b_n}{2b_n}=\frac{c_0}{2}.$$

It follows that

$$\limsup_{r\to+\infty}\frac{1}{r}\int_0^r\|f(t)\|dt\geq\frac{c_0}{2}>0.$$

Thus $f\notin\mathcal{E}(\mathbb{R}^+,X,\lambda)$. Let $r>0$ be sufficiently large. There exists a unique integer n such that $b_n\leq r<b_{n+1}$. As $f(t)=\varphi(t)x_0$ and $\|x_0\|=1$, one has

$$\int_0^r\|f(t)\|dt=\int_0^r\varphi(t)dt.$$

φ vanishes outside $\bigcup_{k\geq 1}[b_k, 2b_k]$, $k \geq 1$. Since $r < b_{n+1}$, the segment $[0,r]$ does not intersect any interval $[b_k, 2b_k]$ with $k \geq n+1$. Consequently, the part of $[0,r]$ where φ can be nonzero is contained in $\bigcup_{k=1}^{n}[b_k, 2b_k]$. As $0 \leq \varphi \leq 1$, one obtains

$$\int_0^r \varphi(t)dt \leq \sum_{k=1}^{n} \lambda([b_k, 2b_k]) .$$

But $\lambda([b_k, 2b_k]) = b_k$. So

$$\int_0^r \|f(t)\|dt \leq \sum_{k=1}^{n} b_k = S_n .$$

As V is increasing and $r \geq b_n$, one has $V(r) \geq V(b_n)$. So

$$\frac{1}{V(r)}\int_0^r \|f(t)\|dt \leq \frac{S_n}{V(b_n)} .$$

By the previous estimate $\frac{S_n}{V(b_n)} \leq \frac{n}{2}$. Thus,

$$0 \leq \frac{1}{V(r)}\int_0^r \|f(t)\|dt \leq \frac{n}{2} .$$

When $r \to +\infty$, we also have $n \to +\infty$. So

$$\lim_{r\to+\infty} \frac{1}{V(r)}\int_0^r \|f(t)\|dt = 0 .$$

Consequently, $f \notin \mathcal{E}(\mathbb{R}^+, X, \lambda, \nu)$; from which it follows that

$$\mathcal{E}(\mathbb{R}^+, X, \lambda) \subsetneq \mathcal{E}(\mathbb{R}^+, X, \lambda, \nu) .$$

The necessary condition is a direct consequence of **Theorem 13**.

Example 31 *For* $n \geq 2$, *set* $a_n = 2^{n^2}$. *Define the positive measure*

$$\nu = \lambda + \sum_{n\geq 2} n a_n \delta_{a_n} , \tag{21}$$

where δ_{a_n} is the Dirac delta function at the point a_n. Thus, for every Borel set $A \subset \mathbb{R}^+$,

$$\nu(A) = \lambda(A) + \sum_{n\geq 2} n a_n \delta_{a_n}(A) .$$

We consider the point $r_n = a_n$. We have $V(a_n) = a_n + \sum_{k=2}^{n} k a_k$. *We have* $V(a_n) > (n+1)a_n$, *and hence* $\frac{a_n}{V(a_n)} \to 0$. *So,* $\liminf_{r\to+\infty} \frac{r}{V(r)} = 0$. *For the* lim sup, *let's put* $r_n = \frac{a_{n+1}}{2}$. *Then, for sufficiently large n,* $a_n < r_n < a_{n+1}$. *So*

$$V(r_n) = r_n + \sum_{k=2}^{n} k a_k \quad \text{and} \quad \sum_{k=2}^{n} k a_k < n^2 a_n = n^2 2^{n^2} = o(2^{(n+)^2}) = o(r_n)$$

then $V(r_n) \sim r_n$. *We conclude that* $\limsup_{r\to+\infty} \frac{r}{V(r)} = 1$. *Thus,*

$$0 = \liminf_{r\to+\infty} \frac{r}{\nu([0,r])} < \limsup_{r\to+\infty} \frac{r}{\nu([0,r])} = 1 < +\infty .$$

4.2 Volterra Operator

Theorem 32 *Let* $\mu, \nu \in \mathcal{M}$, *be such that* (μ,ν) *satisfies* (**HP**) *and* (**H1**). *Let* $f \in PSAP_\omega^{\mu,\nu}(\mathbb{R}^+, X)$, *and let* $(U(t,s))t \geq s$ *be an* ω -periodic, *exponentially stable evolutionary family* (i.e. $i) - \nu)$ *hold*). *Then, the operator* $(Tf)(t) := \int_0^t U(t,s)f(s)ds$ *maps* $\mathcal{P}SAP_\omega^{\mu,\nu}(\mathbb{R}^+, X)$ *into itself.*

For the proof of this theorem, we need the following lemma:

Lemma (Truncation lemma) Assume that the space $\mathcal{E}(\mathbb{R}^+;X;\mu,\nu)$ is bi invariant under positive translations. Let $s>0$ and $h\in CB(\mathbb{R}^+;X)$. Define

$$(\mathcal{R}_s h)(t)=\begin{cases} h(t-s) \text{ for } t\geq s \\ h(0) \text{ for } 0\leq t<s \end{cases}$$

If

$$h\in\mathcal{E}(\mathbb{R}^+;X;\mu,\nu)\ ,$$

then

$$\mathcal{R}_s h\in\mathcal{E}(\mathbb{R}^+,X;\mu,\nu)\ .$$

proof. The function $\mathcal{R}_s h$ is bounded. It is continuous on $[0,s)$ and on $[s,+\infty)$.

Moreover,

$$\lim_{t\to s^-}(\mathcal{R}_s h)(t)=h(0)=(\mathcal{R}_s h)(s)\ .$$

Consequently,

$$\mathcal{R}_s h\in CB(\mathbb{R}^+;X)\ .$$

Let us define the positive translation operator by

$$(T_s q)(t)=q(t+s),\quad t\geq 0\ .$$

For all $t\geq 0$, we have $t+s\geq s$, and it follows that

$$T_s(\mathcal{R}_s h)(t)=(\mathcal{R}_s h)(t+s)=h(t+s-s)=h(t)\ .$$

So,

$$T_s(\mathcal{R}_s h)(t)=h\ .$$

Bi-invariance under translation gives

$$q\in\mathcal{E}(\mathbb{R}^+,X;\mu,\nu)\Leftrightarrow T_s q\in\mathcal{E}(\mathbb{R}^+,X;\mu,\nu)\ .$$

Applying this equivalence with $q=\mathcal{R}_s h$ and by using the identity $T_s(\mathcal{R}_s h)=h$, we obtain

$$\mathcal{R}_s h\in\mathcal{E}(\mathbb{R}^+,X;\mu,\nu)\ .$$

which completes the proof.

We are now in a position to prove **Theorem 32**.

proof. of **Theorem 32** Set $u=Tf$. First of all Tf is continuous and bounded. The boundedness is deduced immediately from

$$\|Tf(t)\|\leq k\|f\|_\infty\int_0^t e^{-\alpha(t-s)}ds\leq\frac{k}{\alpha}\|f\|_\infty$$

Continuity follows from the strong continuity of $(t,s)\to U(t,s)x$, the continuity of f, and Lebesgue dominated convergence theorem (since on the compact set $\{(t,s):0\leq s\leq t\leq R\}$, we have the bound $\|U(t,s)f(s)\|\leq k\|f\|_\infty$).

For $r\geq 0$, we have

$$I(r)=\frac{1}{\nu([0,r])}\int_0^r\|u(t+\omega)-u(t)\|d\mu(t)=$$

$$\frac{1}{\nu([0,r])}\int_0^r\left\|\int_0^{t+\omega}U(t+\omega,s)f(s)ds-\int_0^t U(t,s)f(s)ds\right\|d\mu(t)$$

$$=\frac{1}{\nu([0,r])}\int_0^r\left\|\int_0^{\omega}U(t+\omega,s)f(s)ds+\int_\omega^{t+\omega}U(t+\omega,s)f(s)ds-\int_0^t U(t,s)f(s)ds\right\|d\mu(t)$$

Hence

$$I(r) \le \frac{1}{\nu([0,r])}\int_0^r \left\| \int_0^{\omega} U(t+\omega,s) f(s) ds \right\| d\mu(t) +$$

$$\frac{1}{\nu([0,r])}\int_0^r \left\| \int_0^t U(t+\omega,s+\omega) f(s+\omega) ds - \int_0^t U(t,s) f(s) ds \right\| d\mu(t)$$

By the ω-periodicity of the evolution family, we obtain

$$I(r) \le \frac{1}{\nu([0,r])}\int_0^r \left\| \int_0^{\omega} U(t+\omega,s) f(s) ds \right\| d\mu(t) +$$

$$\frac{1}{\nu([0,r])}\int_0^r \int_0^t \left\| U(t,s)(f(s+\omega)-f(s)) ds \right\| d\mu(t) = I_1(r) + I_2(r)$$

,

where

$$I_2(r) = \frac{1}{\nu([0,r])}\int_0^r \int_0^t \left\| U(t,s)(f(s+\omega)-f(s)) ds \right\| ds d\mu(t) \ .$$

$$I_1(r) = \frac{1}{\nu([0,r])}\int_0^r \left\| \int_0^{\omega} U(t+\omega,\omega) U(\omega,s) f(s) ds \right\| d\mu(t)$$

$$= \frac{1}{\nu([0,r])}\int_0^r \left\| U(t+\omega,\omega) \int_0^{\omega} U(\omega,s) f(s) ds \right\| d\mu(t)$$

$$= \frac{1}{\nu([0,r])}\int_0^r \left\| U(t+\omega,\omega) u(\omega) \right\| d\mu(t)$$

Using the assumptions on the evolutionary family $(U(t,s))_{t\ge s}$, we obtain

$$I_1(r) \le \frac{1}{\nu([0,r])}\int_0^r k e^{-\alpha t} \left\| u(\omega) \right\| d\mu(t) = k \left\| u(\omega) \right\| \frac{1}{\nu([0,r])}\int_0^r e^{-\alpha t} d\mu(t) \ .$$

Since $e^{-\alpha t} \to 0$ as $t \to 0)$, then **Theorem 11 iv**) implies that $I_1(r) = 0$ as $r \to \infty$.

By exponential stability, we obtain

$$I_2(r) = \frac{1}{\nu([0,r])}\int_0^r \int_0^t \left\| U(t,s)(f(s+\omega)-f(s)) \right\| ds d\mu(t)$$

$$\le \frac{k}{\nu([0,r])}\int_0^r \int_0^t e^{-\alpha(t-s)} \left\| f(s+\omega)-f(s) \right\| ds d\mu(t)$$

$$= \frac{k}{\nu([0,r])}\int_0^r \int_0^t e^{-\alpha(t-s)} \varphi(s) ds d\mu(t) = \frac{k}{\nu([0,r])}\int_0^r \int_0^t e^{-\alpha s} \varphi(t-s) ds d\mu(t)$$

,

where $\varphi(s) = \left\| f(s+\omega)-f(s) \right\|$. Then by Fubini's theorem,

$$I_2(r) \le \frac{k}{\nu([0,r])}\int_0^r \int_s^r e^{-\alpha s} \varphi(t-s) d\mu(t) ds \tag{22}$$

Since

$$f \in \mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{R}^+, X) \ ,$$

one has

$$\varphi \in \mathcal{E}(\mathbb{R}^+, \mathbb{R}; \mu, \nu) \ .$$

For any fixed $s \ge 0$, the preceding lemma gives

$$\mathcal{R}_s \varphi \in \mathcal{E}(\mathbb{R}^+, \mathbb{R}; \mu, \nu) \ .$$

For $r > 0$ and $0 \le s \le r$, let's define

$$g_r(s) = \frac{1}{\nu([0,r])}\int_s^r \varphi(t-s) d\mu(t) \ .$$

For $s > r$, set $g_r(s) = 0$. Let $s \geq 0$ be fixed. For $r \geq s$, one has

$$0 \leq g_r(s) = \frac{1}{\nu([0,r])} \int_0^r (\mathcal{R}_s\varphi)(t) d\mu(t) .$$

As $\mathcal{R}_s\varphi$ is positive, we obtain

$$g_r(s) \leq \frac{1}{\nu([0,r])} \int_0^r \mathcal{R}_s\varphi)(t) d\mu(t) .$$

Since $\mathcal{R}_s\varphi \in \mathcal{E}(\mathbb{R}^+, \mathbb{R}; \mu, \nu)$, we obtain

$$g_r(s) \to 0 (r \to +\infty) ,$$

for any fixed $s \geq 0$.

Thanks to (**H1**), there exist $C > 0$ and $r_0 > 0$ such that

$$\frac{\mu([0,r])}{\nu([0,r])} \leq C , \ r \geq r_0 .$$

Therefore, for $r \geq r_0$ and $s \geq 0$,

$$0 \leq g_r(s) \leq \|\varphi\|_\infty \frac{\mu([0,r])}{\nu([0,r])} \leq C \|\varphi\|_\infty .$$

The estimate previously obtained in (22) can then be written in the form

$$I_2(r) \leq k \int_0^{+\infty} e^{-\alpha s} g_r(s) ds .$$

Moreover,

$$0 \leq e^{-\alpha s} g_r(s) \leq C \|\varphi\|_\infty e^{-\alpha s} ,$$

and the function on the right-hand side is integrable over $\mathbb{R}^+$. The dominated convergence theorem therefore implies

$$I_2(r) \to 0 \text{ as } r \to +\infty ,$$

which proves that $u \in \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R}^+, X)$.

4.3 Double measure pseudo $\mathcal{S}$-asymptotically ω-periodic functions with parameters

In what follows, we consider doubly measure pseudo $\mathcal{S}$-asymptotically ω-periodic functions with parameters.

Throughout this subsection, X and Y are Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and $\mathbb{J} \in \{\mathbb{R}^+, \mathbb{R}\}$. The subscript ub means uniformly on bounded subsets of X.

Definition 33. *Let $f : \mathbb{J} \times X \to Y$ be continuous. We say that $f \in SAP_{\omega,\text{ub}}(\mathbb{J} \times X; Y)$ if, for every bounded subset $B \subset X$, the following conditions hold.*

(i) *For every $x \subset B$,*

$$f(\cdot, x) \in CB(\mathbb{J}, Y) .$$

(ii) *For every $\varepsilon > 0$, there exist $\delta > 0$ and a bounded subinterval $I \subset \mathbb{J}$ such that, for all $y \in B$,*

$$\|x - y\|_X \leq \delta \Rightarrow \sup_{t \in \mathbb{J} \setminus I} \|f(t,x) - f(t,y)\|_Y \leq \varepsilon .$$

(iii)

$$\lim_{|t| \to +\infty} \sup_{x \in B} \|f(t+\omega, x) - f(t,x)\|_Y = 0 .$$

Remark Condition (ii) is satisfied, in particular, when the mapping $x \mapsto f(\cdot, x)$ is uniformly continuous on every bounded subset of X, with respect to the supremum norm in the time variable.

Definition 34. *Let $\mu, \nu \in \mathcal{M}$, and let $f : \mathbb{J} \times X \to Y$ be continuous. We say that*

$$f \in \mathcal{PSAP}_{\omega,\text{ub}}^{\mu,\nu}(\mathbb{J} \times X, Y)$$

if, for every bounded subset $B \subset X$, *the following conditions hold.*

(i) *For every* $x \in B$,

$$f(\cdot, x) \in CB(\mathbb{J}, Y) .$$

(ii) *For every* $\varepsilon > 0$, *there exist* $\delta > 0$ *and a bounded subinterval* $I \subset \mathbb{J}$ *such that, for all* $x, y \in B$,

$$\|x - y\|_X \le \delta \Rightarrow \sup_{t \in \mathbb{J} \setminus I} \|f(t,x) - f(t,y)\|_Y \le \varepsilon . \tag{23}$$

(iii) *The scalar-valued function*

$$t \mapsto \sup_{x \in B} \|f(t+\omega, x) - f(t,x)\|_Y \text{ belongs to } \mathcal{E}_e(\mathbb{J}, \mathbb{R}; \mu, \nu) .$$

Remark. In **Definitions** 33 and 34, condition (i) may equivalently be replaced by

$$f(\cdot, 0) \in CB(\mathbb{J}, Y) .$$

4.3.1 Composition Results

Lemma 35. *Let* $f : \mathbb{J} \times X \to Y$ *be continuous and satisfy condition (ii) of Definition 34. Assume that* $f(\cdot, 0) \in CB(\mathbb{J}, Y)$, *and let* $g \in CB(\mathbb{J}, Y)$. *Then the function* $t \mapsto f(t, g(t))$ *is bounded on* $\mathbb{J}$.

proof. The proof follows the first part of the proof of the **Proposition 4.1** in [25], with $\mathbb{J}$ in place of $\mathbb{R}$.

Theorem 36 *Let* $\mu, \nu \in \mathcal{M}$ *satisfy* ***(H1)****. Assume that*

$$f \in \mathcal{PSAP}^{\mu,\nu}_{\omega,\mathrm{ub}}(\mathbb{J} \times X, Y) \text{ and } \varphi \in \mathcal{PSAP}^{\mu,\nu}_{\omega}(\mathbb{J}, X)$$

Then the function

$$[t \mapsto f(t, \varphi(t))] \text{ belongs to } \mathcal{PSAP}^{\mu,\nu}_{\omega}(\mathbb{J}, X) .$$

proof. : 1) By **Lemma 35**, the function $t \to f(t, \varphi(t))$ is bounded.

2) One has

$$\begin{aligned} &f\big(t+\omega, \varphi(t+\omega)\big) - f\big(t, \varphi(t)\big) = f\big(t+\omega, \varphi(t+\omega)\big) - f\big(t, \varphi(t+\omega)\big) \\ &+ f\big(t, \varphi(t+\omega)\big) - f\big(t, \varphi(t)\big) \end{aligned} .$$

Also, one has

$$\big\|f(t+\omega), \varphi(t+\omega)) - f\big(t, \varphi(t+\omega)\big)\big\| \le \sup_{x \in B} \|f(t+\omega, x) - f(t,x)\| ,$$

where B is a closed ball that contains $\varphi(\mathbb{J})$, which implies that

$$t \mapsto f\big(t+\omega, \varphi(t+\omega)\big) - f\big(t, \varphi(t+\omega)\big) \text{ is } (\mu,\nu)\text{-ergodic.} \tag{24}$$

On the other hand, for every $\varepsilon > 0$, there exist $\delta > 0$, and $\exists I$ a bounded interval I such that,

$$\forall x, y \in B,\ \|x - y\| \le \delta \Rightarrow \sup_{t \in \mathbb{J} \setminus I} \|f(t,x) - f(t,y)\| \le \varepsilon .$$

In particular, one has:

$$\forall t \in \mathbb{J} \setminus I,\ \|\varphi(t+\omega) - \varphi(t)\| \le \delta \Rightarrow \big\|f\big(t, \varphi(t+\omega)\big) - f\big(t, \varphi(t)\big)\big\| \le \varepsilon ,$$

then by putting

$$A_{r,\delta} = \big\{t \in \mathbb{J}_r \setminus I,\ \|\varphi(t+\omega) - \varphi(t)\| > \delta\big\} ,$$

and

$$B_{r,\varepsilon} = \big\{t \in \mathbb{J}_r \setminus I,\ \big\|f,\big(t, \varphi(t+\omega)\big) - f\big(t, \varphi(t)\big)\big\| > \varepsilon\big\} ,$$

we obtain $B_{r,\varepsilon} \subset A_{r,\delta}$. Thus

$$0 \le \frac{\mu(B_{r,\varepsilon})}{\nu(\mathbb{J}_r \setminus I)} \le \frac{\mu(A_{r,\delta})}{\nu(\mathbb{J}_r \setminus I)} \underset{r \to +\infty}{\to} 0 ,$$

(since $t \mapsto \varphi(t+\omega)-\varphi(t)$ is $(\mu,\nu)-$ergodic, from which we have

$$\forall_{\varepsilon}>0,\ \frac{\mu(B_{r,\varepsilon})}{\nu(\mathbb{J}_r \setminus I)} \underset{r\to+\infty}{\to} 0,$$

it follows that

$$t \mapsto f\left(t,\varphi(t+\omega)\right)-f\left(t,\varphi(t)\right) \text{ is } (\mu,\nu)-\text{ergodic}. \tag{25}$$

From (24) and (25), we deduce that $t \mapsto f\left(t+\omega,\varphi(t+\omega)\right)-f\left(t,\varphi(t)\right)$ is $(\mu,\nu)-$ergodic, which means that $t \mapsto f\left(t,\varphi(t)\right)$ belongs to $\mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{J},Y)$.

Corollary 37 *Let $\varphi \in \mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{J},Y)$, and let $\Phi: X \to Y$ be a (generally nonlinear) operator which is uniformly continuous on the bounded subsets of X. Then the function $F: \mathbb{J} \to Y$ defined by $F(t)=\Phi\circ\varphi(t)$ is in $\mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{J},Y)$.*

proof. Since φ is bounded, there exists $R>0$ such that $\varphi(\mathbb{J}) \subset \overline{B}(0,R)$. The result is then a direct consequence of **Theorem 36** with $B=\overline{B}(0,R)$ and $f(t,x)=\Phi(x)$.

Proposition 38 *[[25], **Proposition 4.2**] Let $E_1,E_2..,E_n$ and F be Banach spaces. If $u: E_1\times..E_n \to F$ is continuous and n- linear function, then u is uniformly continuous on every bounded set.*

Corollary 39 *Let $E_1,E_2..,E_n$ and F be Banach spaces. If $u: E_1\times.E_n \to F$ is a continuous n-linear function, and $f_i \in \mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{J},E_i)$ then $u(f_1,..,f_n) \in \mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{J},F)$.*

proof. It is a consequence of **Corollary 37** and **Proposition 38**.

Consider the following hypothesis (**H3**). Let $f: \mathbb{J}\times X \to Y$ be continuous.

Assume that $f(\mathbb{J}\times B)$ is bounded for every bounded subset B of X, that f satisfies condition (iii) of the **Definition 34** and f satisfies a time dependent Lipschitz condition in he the second variable, uniformly with respect to the first variable. Specifically, there exists $\ell \in B(\mathbb{J},\mathbb{R},\mu,\nu)$ that is locally integrable on J such that:

$$\left\|f(t,x)-f(t,y)\right\| \le \ell(t)\left\|x-y\right\|,\ x,y\in X,\ t\in\mathbb{J},$$

Theorem 40 *Let $\mu,\nu \in \mathcal{M}$ be such that (μ,ν) satisfies* (**H1**)*, and let $f: \mathbb{J}\times X \to Y$. Assume that the condition **(H3)** is satisfied, and let $u \in \mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{J},X)$. Then the function $f\left(.,u(.)\right)$ belongs to $\mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{J},Y)$.*

proof. : Since $u \in \mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{J},X)$, the function u belongs to $CB(\mathbb{J},X)$. There exists a real number $R>0$ such that $u(\mathbb{J}) \subset \overline{B}_X(0,R)$. For all $t\in\mathbb{J}$, we also have $u(t+\omega)\in\overline{B}_X(0,R)$. By hypothesis (**H3**), the image of $\mathbb{J}\times\overline{B}_X(0,R)$ under f is bounded. Let's set

$$M=\sup\left\{\left\|f(t,x)\right\|_Y : t\in\mathbb{J},\ x\in\overline{B}_X(0,R)\right\}.$$

Then $M<+\infty$. As f and u are continuous, the function $t\mapsto f(t,u(t))$ is continuous, and also bounded, because $\left\|f(t,u(t))\right\|_Y \le M$. For $r>0$, we put

$$D(r)=\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\left\|f\left(t+\omega,u(t+\omega)\right)-f\left(t,u(t)\right)\right\|_Y d\mu(t).$$

By adding and substracting $f(t,u(t+\omega))$, we obtain $D(r)\le L(r)+K(r)$, where

$$L(r)=\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\left\|f\left(t+\omega,u(t+\omega)\right)-f\left(t,u(t+\omega)\right)\right\|_Y d\mu(t)$$

and

$$K(r)=\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\left\|f\left(t,u(t+\omega)\right)-f\left(t,u(t)\right)\right\|_Y d\mu(t).$$

As $u(t+\omega)\in\overline{B}_X(0,R)$, one has

$$L(r)=\frac{1}{\nu(\mathbb{J}_r)}\int_{\mathbb{J}_r}\sup_{x\in\overline{B}_X(o,R)}\left\|f\left(t+\omega,x\right)-f\left(t,x\right)\right\|_Y d\mu(t).$$

The uniform ergodicity condition contained in (**H3**) then yields

$$L(r) \to 0,\ (r \to +\infty)\ .$$

Fix $\varepsilon > 0$ and set

$$A_{r,\varepsilon} = \left\{ t \in \mathbb{J}_r : \left\| u(t+\omega) - u(t) \right\|_X > \varepsilon \right\}\ .$$

On $A_{r,\varepsilon}$, we use the boundedness of f on $\mathbb{J} \times \overline{B}_X(0,R)$:

$$\left\| f\left(t, u(t+\omega)\right) - f\left(t, u(t)\right) \right\|_Y \le C\ ,$$

where $C = 2M$. On $\mathbb{J}_r \setminus A_{r,\varepsilon}$, the Lipschitz condition yields

$$\left\| f\left(t, u(t+\omega)\right) - f\left(t, u(t)\right) \right\|_Y \le \ell(t) \left\| u(t+\omega) - u(t) \right\|_X \le \varepsilon\, \ell(t)\ .$$

Consequently,

$$K(r) = C \frac{\mu(A_{r,\varepsilon})}{\nu(\mathbb{J}_r)} + \varepsilon \frac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \ell(t) d\mu(t)\ .$$

Since $u \in \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{J}, X)$, the function $[t \mapsto u(t+\omega) - u(t)] \in \mathcal{E}(\mathbb{J}, X; \mu, \nu)$. The characterization of ergodic functions therefore implies, for all $\varepsilon > 0$,

$$\frac{\mu(A_{r,\varepsilon})}{\nu(\mathbb{J}_r)} \to 0 \quad (r \to +\infty)\ .$$

Furthermore, since $\ell \in B(\mathbb{J}, \mathbb{R}^+; \mu, \nu)$, the number $N = \limsup_{r\to\infty} \frac{1}{\nu(\mathbb{J}_r)} \int_{\mathbb{J}_r} \ell(t) d\mu(t)$ is finite. By taking the lim sup in the estimate for $K(r)$, we obtain $0 \le \limsup_{r\to\infty} K(r) \le \varepsilon N$. This last inequality holds for all $\varepsilon > 0$. Letting ε tend to zero, we obtain $\limsup_{r\to\infty} K(r) = 0$. As $K(r) \ge 0$, we conclude that

$$K(r) \to 0\ \ (r \to +\infty)\ .$$

Finally,

$$0 \le D(r) \le L(r) + K(r)\ ,$$

and it follows

$$D(r) \to 0\ \ (r \to +\infty)\ .$$

Thus, the $\left[t \mapsto f\left(t, u(t)\right) \right] \in \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{J}, Y)$. The proof is complete.

Theorem 41 *If (**HP**) and (**H1**) hold, then* $\mathcal{E}(\mathbb{J}, X, \mu, \nu) \subset \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{J}, X)$ *for all* $\omega > 0$.

proof. Let $f \in \mathcal{E}(\mathbb{J}, X, \mu, \nu)$. By **Theorem 18**, we have $\mathcal{E}(\mathbb{J}, X, \mu, \nu)$ is translation invariant, so $f_\omega \in \mathcal{E}(\mathbb{J}, X, \mu, \nu)$. It follows that $f_\omega - f \in \mathcal{E}(\mathbb{J}, X, \mu, \nu)$.

Consequently, $f \in \mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{J}, X)$. This completes the proof.

5. Applications to Evolution Equations

Now, consider equation (1).

In the sequel, we make the following assumptions:

h₁) $A(t)$ generates an ω-periodic (ω>0) exponentially stable evolutionary family in X.

h₂) Let $f : \mathbb{R}^+ \times X \to X$ be continuous. Assume that, $f(\mathbb{R}^+ \times B)$ is bounded for every bounded subset B of X, that f satisfies condition (iii) of the **Definition 34** and that f satisfies a time dependent Lipschitz condition in the second variable, uniformly with respect to the first variable. Specifically, there exists $\ell \in B(\mathbb{R}^+, \mathbb{R}^+, \mu, \nu)$ that is locally Lebesgue integrable on $\mathbb{R}^+$ such that

$$\left\| f(t,x) - f(t,y) \right\| \le \ell(t) \left\| x - y \right\|,\ \ x, y \in X,\ \ t \ge 0\ , \tag{26}$$

and

$$\limsup_{t\to\infty}\int_0^t e^{-\alpha(t-s)}\ell(s)ds<\frac{1}{k}. \tag{27}$$

Definition 42 A continuous function $x:\mathbb{R}^+\to X$ is called a mild solution of (1) if

$$x(t)=U(t,0)x_0+\int_0^t U(t,s)f(s,x(s))ds \text{ for } t\geq 0 .$$

Theorem 43 *Let $\mu,\nu\in\mathcal{M}$ be such that (μ,ν) satisfies **(HP)** and **(H1)**. Under assumptions **h_1)-h_2)**, equation (1) possesses a unique mild solution in $\mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R}^+,X)$.*

proof. Consider the mapping T defined on $\mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R}^+,X)$ by

$$Tu(t):=U(t,0)x_0+\int_0^t U(t,s)f(s,u(s)ds) . \tag{28}$$

By **Theorem 32** and **Theorem 40,**

$$\left[t\to\int_0^t U(t,s)f(s,u(s))ds\right]\in\mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R}^+,X).$$

Moreover, property **(v) in Definition 1** yields the following inequality:

$$\left\|U(t,0)x_0\right\|\leq ke^{-\alpha t}\left\|x_0\right\|$$

Then $\left[t\mapsto U(t,0)x_0\right]\in C_0(\mathbb{R}^+,X)\subset\mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R}^+,X)$ (By Theorem 11). Thus T is well-defined. Now, for all $u,v\in\mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R}^+,X)$, and all $t\in\mathbb{R}^+$

$$\left\|(Tu)(t)-(Tv)(t)\right\|\leq 0\int_0^t ke^{-\alpha(t-s)}\ell(s)\left\|u(s)-v(s)\right\|ds . \tag{29}$$

The condition (27) implies the existence of $b>0$ such that

$$M:=\sup_{t\geq b}\int_0^t e^{-\alpha(t-s)}\ell(s)ds<\frac{1}{k}. \tag{30}$$

Consider the function $\ell_b(t)=\ell(t)1_{[0,b]}(t)$. Then since $\ell\in L^1_{loc}(\mathbb{R}^+)$, we have $\ell_b\in L^1(\mathbb{R}^+)$. Let $c>0$. Define an equivalent norm on $\mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R}^+,X)$ (by

$$N(f)=\sup_{t\in\mathbb{R}^+}e^{-cL(t)}\left\|f(t)\right\|, \text{ where } c>0 \text{ and } L(t)=\int_0^t\ell_b(s)ds .$$

The function L is absolutely continuous on $\mathbb{R}^+$, and we have $\left(e^{-c\|\ell_b\|_1}\left\|f\right\|_\infty\leq N(f)\leq\left\|f\right\|_\infty\right)$.

Hence $\mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R}^+,X)$ equipped with N is a Banach space. From (29), we have for all $t\in\mathbb{R}^+$

$$\begin{aligned}
&\left\|(Tu)(t)-(Tv)(t)\right\|\leq kN(u-v)\int_0^t e^{-\alpha(t-s)}\ell(s)e^{cL(s)}ds\\
&\leq kN(u-v)\left(\int_0^t e^{-\alpha(t-s)}\ell_b(s)e^{cL(s)}ds+\int_0^t e^{-\alpha(t-s)}\ell(s)1_{\mathbb{R}^+\setminus[0,b]}e^{cL(s)}ds\right)\\
&\leq kN(u-v)\left(\underbrace{\int_0^t\ell_b e^{cL(s)}ds}_{I_1(t)}+\underbrace{e^{cL(t)}\int_0^t e^{-\alpha(t-s)}\ell(s)1_{\mathbb{R}^+\setminus[0,b]}(s)ds}_{I_2(t)}\right)
\end{aligned}$$

On the one hand, since L is absolutely continuous and $L'(t)=\ell_b(t)$ *a.e.*, we have

$$I_1(t)=\frac{1}{c}\left(e^{cL(t)}-1\right)\leq\frac{e^{cL(t)}}{c}.$$

On the other hand, we have for all $t\in\mathbb{R}^+$

$$I_2(t)\leq e^{cL(t)}M . \text{ (Since } I_2(t)=0 \text{ for } t<b .)$$

Hence for all $u,v\in\mathcal{PSAP}_\omega^{\mu,\nu}(\mathbb{R}^+,X)$.

$$N(Tu-Tv)\leq\left(\frac{k}{c}+kM\right)N(u-v) .$$

By (**27**), $kM<1$. Choosing $c>\frac{k}{1-kM}$ makes the nonlinear operator T is a contraction, so it has a unique fixed point. Therefore, Equation (1) has a unique mild solution in $\mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{R}^+,X)$.

Theorem 44 Condition **(27) in h_2) holds if at least** one of the following five conditions is satisfied:

1) $\|\ell\|_{L^\infty}<\frac{\alpha}{k}$.

2) $\|\ell\|_{S^1}<\frac{1-e^{-\alpha}}{k}$, where $\|\ell\|_{S^1}:=\sup_{t\in\mathbb{R}+}\int_t^{t+1}\ell(s)\,ds<\infty$.

3) $\lim_{t\to\infty}\int_t^{t+1}\ell(s)\,ds=0$.

4) $\ell\in L^p(\mathbb{R}^+)$, with $1\le p<\infty$.

5) $\lim_{t\to\infty}\ell(t)=0$.

proof. 1) This follows from the inequality:

$$k\sup_{t\in\mathbb{R}+}\int_t^{t+1}e^{-\alpha(t-s)}\ell(s)\,ds\le k\|\ell\|_{L\infty}\sup_{t\in\mathbb{R}+}\int_0^t e^{-\alpha(t-s)}d(s)\ =\frac{k\|\ell\|_{L\infty}}{\alpha}<1\ .$$

2) Assume that the second condition holds, that is

$$\|\ell\|_{S^1}<\frac{1-e^{-\alpha}}{k},\quad \text{where}\quad \|\ell\|_{S^1}=\sup_{\alpha\ge0}\int_\alpha^{\alpha+1}\ell(v)dv\ .$$

For $t\ge0$, let's put

$$I(t)=\int_0^t e^{-\alpha(t-s)}\ell(s)ds\ .$$

Let us make the change of variables $u=t-s$. Then, one has $s=t-u$ and $ds=-du$. It follows that

$$I(t)=\int_0^t e^{-\alpha u}\ell(t-u)du\ .$$

Now, let $t\ge1$. Let n denote the unique integer such that $n\le t<n+1$. We decompose the integral as follows

$$I(t)=\sum_{j=0}^{n-1}\int_j^{j+1}e^{-\alpha u}\ell(t-u)du+\int_n^t e^{-\alpha u}\ell(t-u)du\ .$$

For every integer j such that $0\le j\le n-1$, one has

$$\int_j^{j+1}e^{-\alpha u}\ell(t-u)du\le e^{-\alpha j}\int_j^{j+1}\ell(t-u)du\ .$$

In the next integral, set $v=t-u$. We obtain

$$\int_j^{j+1}\ell(t-u)du=\int_{t-j-1}^{t-j}\ell(v)dv\le\|\ell\|_{S^1}\ .$$

Thus

$$\int_j^{j+1}e^{-\alpha u}\ell(t-u)du\le e^{-\alpha j}\|\ell\|_{S^1}\ .$$

For the last part of the integral, we obtain

$$\int_n^t e^{-\alpha u}\ell(t-u)du\le e^{-\alpha n}\int_n^t\ell(t-u)du\ .$$

Using the change of variables $v=t-u$ again, we obtain

$$\int_n^t\ell(t-u)du=\int_0^{t-n}\ell(v)dv\ .$$

As $0\le t-n<1$, it follows that

$$\int_0^{t-n}\ell(v)dv\le\|\ell\|_{S^1}\ .$$

Consequently,

$$\int_n^t e^{-\alpha u}\ell(t-u)du \le e^{-\alpha n}\left\|\ell\right\|_{S^1}.$$

By combining the previous estimates, we obtain

$$I(t) \le \left\|\ell\right\|_{S^1}\sum_{j=0}^{n} e^{-\alpha j}.$$

The geometric series yields

$$I(t) \le \frac{\left\|\ell\right\|_{S^1}}{1-e^{-\alpha}}.$$

Thus,

$$\limsup_{t\to+\infty}\int_0^t e^{-\alpha(t-s)}\ell(s)ds \le \frac{\left\|\ell\right\|_{S^1}}{1-e^{-\alpha}}.$$

By the second condition

$$\frac{\left\|\ell\right\|_{S^1}}{1-e^{-\alpha}} < \frac{1}{k}.$$

Then, we obtain

$$\limsup_{t\to+\infty}\int_0^t e^{-\alpha(t-s)}\ell(s)ds < \frac{1}{k}.$$

This proves the required inequality.

3) One has

$$I(t) = \int_0^t e^{-\alpha(t-s)}\ell(s)ds = e^{-\alpha t}\int_0^t e^{\alpha s}\ell(s)ds \le e^{-\alpha t}\int_0^{\lfloor t\rfloor+1} e^{\alpha s}\ell(s)ds$$
$$\le e^{-\alpha t}\sum_{j=0}^{\lfloor t\rfloor}\int_j^{j+1} e^{\alpha s}\ell(s)ds \le e^{-\alpha t}\sum_{j=0}^{\lfloor t\rfloor} e^{\alpha(j+1)}\int_j^{j+1}\ell(s)ds = e^{-\alpha t}o\left(\sum_{j=0}^{\lfloor t\rfloor} e^{\alpha(j+1)}\right)$$
$$= e^{-\alpha t}o\left(\sum_{j=0}^{\lfloor t\rfloor} e^{\alpha j}\right) = e^{-\alpha t}\frac{e^{\alpha(\lfloor t\rfloor+1)}-1}{e^{-\alpha}-1}o(1) \underset{t\to+\infty}{\to} 0 < \frac{1}{k}$$

To justify the last result, we can use the following classical result on the series with positive terms:

If $u_n = o(v_n)$ and $\sum_{n\ge 0} v_n$ diverges, then $\sum_{j=0}^{n} u_j = o\left(\sum_{j=0}^{n} v_j\right)$.

4) It is clear that if $\ell \in L^p(\mathbb{R}^+)$ with $p=1$, then $\lim_{t\to\infty}\int_t^{t+1}\ell(s)ds = 0$. The same conclusion holds for $p>1$ by Hölder's inequality

$$\int_t^{t+1}\ell(s)ds \le \left(\int_t^{t+1}\left(\ell(s)\right)^p ds\right)^{\frac{1}{p}}.$$

Hence, in both cases, one can apply 3).

5) $\lim_{t\to\infty}\ell(t) = 0$ implies $\lim_{t\to\infty}\int_t^{t+1}\ell(s)ds = 0$, hence one can apply 3).

Consequently, (**27**) is satisfied. Therefore, by **Theorem 43** equation (1) has a unique mild solution in $\mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{R}^+, X)$.

Remark 45 ***Theorem 43*** *covers the case of semilinear equations where the linear part is the infinitesimal generator of a semigroup which is exponentially stable. Consider the equation* ***(1)****, where* $x_0 \in X$ *and* $A: D(A) \to X$ *is the infinitesimal generator of a semigroup* $\left(T(t)\right)_{t\ge 0}$. *If the semigroup is exponentially stable, with* $\left\|T(t)\right\| \le ke^{-\alpha t}$ *for all* $t \ge 0$, *and* f *satisfies h2) and* (μ,ν) *satisfies* ***(HP)*** *and* ***(H1)****, then Equation* ***(1)*** *possesses a unique mild solution in* $\mathcal{PSAP}_{\omega}^{\mu,\nu}(\mathbb{R}^+, X)$. *This last result is a corollary of* ***Theorem 43*** *by setting* $U(t,s) = T(t-s)$ *for* $t \ge s$.

Let us recall the following proposition which is needed in what follows.

Proposition 46 *[[26], page 37] Let* $X = L^2(\mathbb{R}^m)$, $A = \Delta$, *and* $D(A) = W^{2,2}(\mathbb{R}^m)$. *The operator* A *generates a contraction semigroup on* X *and it is self-adjoint. Moreover, its graph norm is equivalent to that of* $W^{2,2}(\mathbb{R}^m)$.

5.1 Illustrative Example

Example 47 *In this subsection, we apply* ***Theorem 43*** *with* $\mu = \lambda$ *(where* λ *is the Lebesgue measure and* ν *is the measure defined by 21), to establish the existence of a* (λ, ν) S *-asymptotically* ω *-periodic solution. Let us consider the following reaction-diffusion model with time-dependent diffusion coefficients given by:*

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \delta(t)\dfrac{\partial^2}{\partial x^2}u(t,x) + d(t)u(t,x) + \left[f\left(t,u(t;.)\right)\right](x),\ t \in \mathbb{R}^+, \\ u(0,.) = u_0 \in L^2(\mathbb{R},\lambda) \end{cases} \tag{31}$$

where $\delta; d : \mathbb{R} \to \mathbb{R}$ *are continuous w-periodic functions and there exist* d_0 , $\delta_0 > 0$ *such that* $\delta(t) > \delta_0$ *and* $d(t) \le -d_0 \forall t \in \mathbb{R}^+$.

Equation (31) is understood as an abstract evolution equation in $X = L^2(\mathbb{R})$ *and its solutions are considered in the mild sense.*

The abstract formulation of the problem (31).

In order to establish the existence and uniqueness of the (λ, ν) *-pseudo* $\mathcal{S}$ *-asymptotically* ω *-periodic solutions to our model, given by Equation (31), we give its associated abstract form:*

$$u'(t) = A(t)u(t) + f(t,u(t));\ t \in \mathbb{R}^+ . \tag{32}$$

Here $u(t)$ *denotes the mapping* $u(t) = u(t,.) : x \to u(t,x)$. *First,* ***Proposition 29*** *and* ***example 31*** *show that* (***HP***) *and* (***H1***) *are satisfied.*

$$f : \mathbb{R}^+ \times X \to X .$$

More precisely, the function used in the example is defined by

$$f(t,\phi) = h(t)g(\phi) + b(t),\ t \ge 0,\ \phi \in X ,$$

where $b \in \mathcal{PSAP}_{\omega}^{\lambda,\nu}(\mathbb{R}^+, X)$, $g : X \to X$ *is Lipschitz continuous on* X *with Lipshitz constant* ℓ_g *and* $h(t) = \exp\left(-t^2 \left|\sin(\pi t)\right|\right)$. *Thus, all time-dependent functions are considered on the same time domain as the equation. If the coefficients are initially defined on R, only their restrictions to* $\mathbb{R}^+$ *are used in the example. We set*

$$X = L^2(\mathbb{R}),\ D(A) = H^2(\mathbb{R}) ,$$

where $H^2(\mathbb{R})$ *is the usual Sobolev space, and for every* $\phi \in D(A)$, $A\phi = \dfrac{d^2\phi}{dx^2}$. *Thus,*

$$A : D(A) \subset X \to X$$

is the usual realization of the one-dimensional Laplacian in $L^2(\mathbb{R})$. *Since the spatial domain is the whole real line, no boundary condition is imposed.*

Finally, the family of operators is given by

$$A(t) = \delta(t)A + d(t)I, \quad t \ge 0 , \ with\ \ D(A(t)) = D(A) = H^2(\mathbb{R}) .$$

It is well known that $(A, D(A))$ *generates a bounded strongly continuous semi- group* $(T(t))_{t \ge 0}$ *on* X , *i. e.,* $\left\|(T(t))\right\| \le M$ *(see [26], p. 37). Now, clearly, the operator*

$$A(t) := \delta(t)A + d(t)I \ \ with\ \ D(A(t)) = D(A);\ t \in \mathbb{R}^+$$

generates the strongly continuous evolution family

$$U(t+\omega, s+\omega) = U(t,s) = \exp\left(\int_s^t d(\sigma)d\sigma\right) T\left(\int_s^t \delta(\tau)d\tau\right),\ t \ge s .$$

Note that the formula $T\left(\int_s^t \delta(\tau)d\tau\right)$, $t \ge s$, *corresponds to the mild solution for Equation (31) with* $d = f = 0$. *Moreover, we have*

$$\left\|U(t,s)\varphi\right\| \le Me^{-d_0(t-s)}\left\|\varphi\right\|,\ t \ge s,\ \varphi \in X .$$

Therefore, hypothesis $\mathbf{h_1}$ *is satisfied.*

Let us prove that $h \in L^1(\mathbb{R}^+,\mathbb{R}) \bigcap \mathcal{E}(\mathbb{R}^+,\mathbb{R},\lambda,\nu)$ *. Indeed*

$$\int_n^{n+1} |\, h(t)\,|\, dt = \int_0^1 e^{-(t+n)^2 \sin(\pi t)} dt \le \int_0^1 e^{-n^2 \sin(\pi t)} dt$$

$$= 2\int_0^{\frac{1}{2}} e^{-n^2 \sin(\pi t)} dt \le 2\int_0^{\frac{1}{2}} e^{-2n^2 t} dt = O\left(\frac{1}{n^2}\right). \quad \text{Thus } h \in L^1(\mathbb{R}^+, \mathbb{R})$$

Therefore, the fourth condition in **Theorem 44** *is satisfied, so relation (27) in* $\mathbf{h_2}$ *holds.*

Since $\mu = \lambda$ *is the Lebesgue measure,* $L^1(\mathbb{R}^+,\mathbb{R},\lambda) \subset \mathcal{E}(\mathbb{R}^+,\mathbb{R},\lambda,\nu)$ *. Consequently* $h \in \mathcal{E}(\mathbb{R}^+,\mathbb{R},\lambda,\nu)$ *.*

On the other hand

$$\begin{aligned}
&\left\| f(t,\varphi) - f(t,\psi) \right\|_X \le \left\| h(t)(g(\varphi) - g(\psi)) \right\|_X + \left\| b(t+\omega) - b(t) \right\|_X \\
&\le \left\| h(t)(g(\varphi) - g(\psi)) \right\|_X + \left\| b(t+\omega) - b(t) \right\|_X \\
&\le h(t)\,|\,\ell_g \left\| \varphi - \psi \right\|_X + \left\| b(t+\omega) - b(t) \right\|_X = \ell(t) \left\| \varphi - \psi \right\|_X + \left\| b(t+\omega) - b(t) \right\|_X ,
\end{aligned}$$

where

$$[t \mapsto \ell(t) = l_g h(t)] \in \mathcal{E}(\mathbb{R}^+,\mathbb{R},\lambda,\nu) \subset B(\mathbb{R}^+;\mathbb{R},\lambda,\nu) \ ,$$

moreover one has

$$\left\| f(t+\omega,\varphi) - f(t,\varphi) \right\|_X \le |\, h(t+\omega) - h(t)\,| \left\| g(\varphi) \right\|_X + \left\| b(t+\omega) - b(t) \right\|_X .$$

$$\sup_{\varphi \in B} \left\| g(\varphi) \right\|_X \le C \quad \textit{(for some constant C).}$$

In fact, B *is contained in some closed ball* $\overline{B}(0,R)$ *.*

$$|\, g(\varphi)\,| \le |\, g(\varphi) - g(0)\,| + |\, g(0)\,| \le \ell_g \left\| \varphi \right\| + |\, g(0)\,| \le C \ ,$$

where $C = \ell_g R + |\, g(0)\,|$ *.*

$$\begin{aligned}
&\frac{1}{\nu([0,r])} \int_0^r \sup_{\varphi \in B} \left\| f(t+\omega,\varphi) - f(t,\varphi) \right\|_X d\lambda(t) \\
&\le \frac{C}{\nu([0,r])} \int_0^r |\, h(t+\omega) - h(t)\,| d\lambda(t) + \frac{1}{\nu([0,r])} \int_0^r \left\| b(t+\omega) - b(t) \right\|_X d\lambda(t) \\
&\le \frac{2C \left\| h \right\|_{L^1(\mathbb{R}+)}}{\nu([0,r])} + \frac{1}{\nu([0,r])} \int_0^r \left\| b(t+\omega) - b(t) \right\|_X d\lambda(t)
\end{aligned} ,$$

therefore

$$\lim_{r \to \infty} \frac{1}{\nu([0,r])} \int_0^r \sup_{\varphi \in B} \left\| f(t+\omega,\varphi) - f(t,\varphi) \right\|_X d\lambda(t) = 0 \ ,$$

hence condition (iii) of the **Definition 34** *holds. On the other hand, since g is Lipschitz continuous, therefore it maps every bounded set to a bounded set. Thus, for every bounded subset* B *of* X, $g(B)$ *is bounded. Since h and b are also bounded,* $f(\mathbb{R}^+ \times B)$ *is bounded. Hence, hypothesis (*$\mathbf{h_2}$*) holds, and all assumptions of* **Theorem 43** *are satisfied. Hence we have the following main result: equation (32) has a unique mild solution which belongs to* $\mathcal{PSAP}_\omega^{\lambda,\nu}(\mathbb{R}^+, X)$ *.*

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Informed Consent

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Both authors contribute equally to the writing of this paper. Both authors have read and approved the submitted manuscript.

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